TEACHING / TRAINING MODULE
MATHEMATICS

CLASS-XI & XII

Compiled by:

Nilambar Biswal
Reader in Mathematics (Retd.)

Soumyendra Mishra
Sr Lecturer in Mathematics
P.N. College (Auto), Khordha

SCHEDULED CASTES & SCHEDULED TRIBES RESEARCH AND TRAINING INSTITUTE (SCSTRTI)
SC & ST DEVELOPMENT DEPARTMENT
GOVERNMENT OF ODISHA
BHUBANESWAR

September 2018

FOREWORD

The ST and SC Development Department, Government of Odisha, has initiated an innovative effort by setting up an ***Academic Performance Monitoring Cell*** (APMC) in Scheduled Castes and Scheduled Tribes Research and Training Institute (SCSTRTI) to monitor the Training and Capacity Building of teachers of SSD Higher Secondary Schools and Ekalabya Model Residential Schools (EMRS) under the administrative control of the ST & SC Development Departme. This innovative program is intended to ensure quality education in the Higher Secondary Level of the schools of the ST & SC Development Department.

The modules and lesson plans are prepared for the +2 Science and Commerce stream' in all the subjects such as Physics, Chemistry, Botany, Zoology, Mathematics, Information Technology, Odia, English and Commerce for both the years in line with the syllabus of Council of Higher Secondary Education (CHSE).

These modules/lesson plans are self contained. The subiect experts who are the best in their respective subjects in the State have been roped in for the exercise. They have given their precious time to make the module as activity based as possible.

I hope, this material will be extremely useful for the subject teachers in effective class room transactions and will be helpful in improving the quality education at the Higher Secondary Level. I also take this opportunity to thank all the subject experts of different subjects for rendering help and assistance to prepare the modules/lesson notes and lesson plans within a record time.

Prof. (Dr.) A.B.Ota
Director and Special Secretary,
SCSTRTI

CONTENTS

Module

ABSTRACT

Set :

Preliminary idea on set, Set representation, Empty set, Singleton set, Finite and Infinite set, Some standard sets like N, Z, Q, R, ℂ etc., Cardinality, Equivalent sets, Interval, Subset, Superset, Power set, Universal set, Complement of a set, Difference of sets, Set Operations – Union, Intersection, Symmetric difference, De Morgan's laws, Cartesian product of sets. Ordered pair, $R \times R,\ R^n$.

Relation :

Definition of relation, Domain, Co-domain and Range of a relation, How to represent a relation, Idea about one-one, many-one, one-many, many-many relation with examples, Reflexive relation, Symmetric relation, Transitive relation, Equivalence relation, Congruence Modulo Relation on integers, Anti symmetric relation, Partial ordering relation, Total ordering relation, How an equivalence relation splits a set into disjoint equivalence classes.

Function : Definition of function, Domain, Co-domain and Range of a function, Examples of some standard functions like – Constant function, Polynomial function, Algebraic function, Rational function, Modulus function, Signum function, Exponential function, Logarithmic function, Greatest integer function with related graphs. Idea about Into function, Onto function and Bijective function. Sum, Difference, Product and Quotient of functions.

Trigonometric Functions :

Positive and Negative angle of measurement, System of angle measurement – Centesimal, Sexagesimal, Circular, Angle measurement in Co-ordinate plane, Standard trigonometric functions like $f(x) = sinx,\ f(x) = cosx, \dots etc$ with diagram and respective domain and range.

Different formulas for compound and multiple angles like :-

$\sin(x \pm y)$, $\cos(x \pm y)$, $\tan(x \pm y)$, … $etc\ and\ \ sin2x, cos2x, tan2x, \dots\ etc,$

and $sin3x, cos3x, tan3x, \dots etc$. Formulas for $\sin C\ \pm sin\ D$ and

$\cos C\ \pm \cos D$. Solution of Trigonometric Equations. Principal Solution, General Solution. Properties of triangle – Sine, Cosine formulae.

Principle of Mathematical Induction :

(i) Procedure of proof by Mathematical Induction

(ii) Application of Induction Method

(iii) Related problems

Complex Number :

What is complex number, Representation of complex number, Algebraic properties of complex numbers, Argand plane and polar representation of complex number, Square root and Cube root of unity, Square root of complex number, De Moivre's theorem to find out **nth** root of a complex number.

Linear Inequalities :

Linear inequalities, Algebraic solution of linear inequalities in one variable and their representation in Real line, Graphical solution of liner inequalities in two variables, Graphical solution of System of linear inequalities in two variables.

Permutation and Combination :

Principle of Counting, Use of factorial $(n!)$ for counting purpose, Permutation formula $P(n,r) = \frac{n!}{(n-r)!}$, Combination formula $C(n,r) = \frac{n!}{r!(n-r)!}$

Binomial Theorem :

Statement and proof of Binomial theorems for +ve integral power, Pascal Triangle, Formula to find middle term(s) and general term in Binomial expansion, Binomial expansions like :-

$(1+x)^{-1}, (1-x)^{-1}, (1+x)^{-2}, (1-x)^{-2}$

Sequence and Series :

(i) General introduction on sequence and series, A.P (Arithmetic Progression),

G.P (Geometric Progression), Partial sum, Arithmetic series and Geometric

series, Way to find sum of infinite series.

(ii) Arithmetico-Geometric Series

(iii) Exponential Series

(iv) Logarithmic Series

(v) Trigonometric Series

Straight Line :

Introduction on Co-ordinate Geometry, Brief discussion on 2D and 3D, 2-Dimensional Cartesian co-ordinate axis system, Distance and Division formulae, Slope of a line, Angle between two lines, Equation of a straight line in different form – Slope-intercept form, Point-slope form, 2-point form, … etc, General equation of a line, Point of intersection, Family of lines, Distance of a point form a line, Shifting of Origin (Translation), Shifting of Axes through rotation.

Conic section :

Locus, Section of a cone, Circle, Ellipse, Parabola, Hyperbola, General equation of a circle, equation of its tangent and normal, Intersection of two circles and angle thus generated, Radical axis.

Equation of a parabola, Equation of tangent and normal.

Equation of an ellipse, Equation of tangent and normal.

Equation of hyperbola, Equation of tangent and normal.

Short-cut method to find equation of tangent and normal.

Eccentricity, Rectangular Hyperbola, Conjugate Hyperbola.

Introduction to 3D :

Discussion on 3-dimensional co-ordinates, Distance formula, Division formula.

Limit :

Idea on limit, Definition of limit i.e. $\lim_{x \to a} f(x) = l$, Some standard formulae of limits. Algebra of limit with related problems, Continuity of a function at a point and in an internal, Different types of discontinuity.

Derivative :

Introduction, Geometrical meaning of derivative, Derivative of a function by ab-initio rule, Derivative of some standard function like

$f(x) = c,\ f(x) = e^x, f(x) = a^x\ , f(x) = logx, f(x) = sinx, f(x) = cosx$ etc. Algebra of derivative, Chain rule.

Mathematical Reasoning :

Introduction on sentence and statement, Truth value of a statement, Connectives and compound statement, Negation, Conjunction, Disjunction, Implication, Both side implication statements with truth table and suitable examples, Converse, Inverse, Contrapositive statements of a given implication, Equivalent statements, Tautology, Fallacy, Quantifier and Predicates.

Statistics :

Introduction, How to collect data, Sample, Population, grouped and ungrouped data, Arithmetic mean, Geometric mean, Harmonic mean, Median, Mode, Measures of dispersion, Mean deviation, Standard deviation, Variance, Co-efficient of variation.

Probability :

Introduction, Basic concepts of probability, Sample space, Outcome, Event, Probability of a particular event, Sure event, Impossible event, Fundamental theorems of probability, Axiomatic approach to probability, Finite probability space, Non-uniform space, Mutually exclusive events, Independent events, Conditional probability, Random variable and its use.

***

LESSON PLAN

(For +2 1st Year)

Unit I	Topic To Be Covered	Number of Classes Required
Set	Preliminary idea on set, Set representation	1
	Empty set, Singleton set, Finite and Infinite set, Some standard sets like N, Z, Q, R, ℂ etc., Cardinality	1
	Equivalent sets, Interval, Subset, Superset, Power set, Universal set, Complement of a set, Difference of sets	1
	Set Operations - Union, Intersection, Symmetric difference, De Morgan's laws, Cartesian product of sets	1
	Doubt Clearing Class	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
Relation	Definition of relation, Domain, Co-domain and Range of a relation, How to present a relation	1
	Idea about one-one, many-one, one-many, many-many relation with examples, Reflexive relation, Symmetric relation, Transitive relation, Equivalence relation	1
	Congruence Modulo relation on integers, Anti symmetric relation, Partial ordering relation, Total ordering relation	1
	How an equivalence relation splits a set into disjoint equivalence classes	1
	Doubt Clearing Class	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1

Function	Definition of function, Domain, Co-domain and Range of a function	1

Unit I	**Topic To Be Covered**	**Number of Classes Required**
Function	Examples of some standard functions like - Constant function, Polynomial function, Algebraic function, Rational function, Modulus function, Signum function	1
	Exponential function, Logarithmic function, Greatest integer function with related graphs)	1
	Idea about Into function, Onto function and Bijective function. Sum, Difference, Product and Quotient of functions	1
	Doubt Clearing Class	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
Trigonometric Functions	Positive and Negative angle of measurement, System of angle measurement - Centesimal, Sexagesimal and Circular	1
	Angle measurement in Co-ordinate plane, Standard trigonometric functions like $f(x) = sinx, f(x) = cosx, \dots etc$ with diagram and respective domain and range	1
	Different formulas for compound and multiple angles like $\sin(x \pm y), \cos(x \pm y), \tan(x \pm y), \dots\ etc\ and$ $sin2x, cos2x, tan2x, \dots\ etc$,and $n3xcos3x, tan3x, \dots etc$. Formulas for $\sin C \pm sin\ D$ and $\cos C \pm \cos D$.	1
	Solution of Trigonometric equations. Principal solution, General solution.	1
	Properties of triangle - Sine, Cosine formulae	1
	Doubt Clearing Class	1

	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
	Problem	1
	Problem	1
	Problem	1

Unit II	**Topic To Be Covered**	**Number of Classes Required**
Principle of Mathematical Induction	Procedure of proof by Mathematical Induction, Application of Induction Method (Related problems)	2
Complex Number	What is complex number, Representation of complex number, Algebraic properties of complex numbers, Argand plane and polar representation of complex number	1
	Square root and Cube root of unity, Square root of complex number, De Moiver's theorem to find out nth root of a complex number	2
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
Linear Inequalities	Linear inequalities, Algebraic solution of linear inequalities in one variable and their representation in Real line	1
	Graphical solution of liner inequalities in two variables, Graphical solution of System of linear inequalities in two variables	1

	Doubt Clearing Class	1
	Problem	1
Permutation and Combination	Principle of Counting, Use of factorial $(n!)$ for counting purpose, Permutation formula $P(n,r) = \frac{n!}{(n-r)!}$, Combination formula $C(n,r) = \frac{n!}{r!(n-r)!}$	2
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
Unit II	**Topic To Be Covered**	**Number of Classes Required**
Binomial Theorem	Statement and proof of Binomial theorems for +ve integral power, Pascal Triangle, Formula to find middle term(s) and general term in binomial expansion,	1
	Binomial expansions like $(1+x)^{-1}, (1-x)^{-1}, (1+x)^{-2}, (1-x)^{-2}$	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
Sequence and Series	General introduction on sequence and series, A.P (Arithmetic Progression), G.P (Geometric Progression), Partial sum, Arithmetic series and Geometric series	2
	Way to find sum of infinite series Arithmetico-Geometric Series, Exponential Series, Logarithmic Series, Trigonometric Series	2
	Doubt Clearing Class	1

	Problem	1
	Problem	1
	Problem	1
Unit III	**Topic To Be Covered**	**Number of Classes Required**
Straight Line	Introduction on Co-ordinate Geometry, Brief discussion on 2D and 3D, 2-Diensional Cartesian co-ordinate axis system, Distance and division formulae	1
	Slope of a line, Angle between two lines, Locus, Equation of a straight line in different form – Slope-intercept form, Point-slope form, 2-point form, … etc	1
	General equation of a line, Point of intersection, Family of lines, Distance of a point form a line	1

Unit III	**Topic To Be Covered**	**Number of Classes Required**
	Shifting of Origin (Translation), Shifting of Axes through rotation	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
	Problem	1
Conic section	Section of a cone, Circle, Ellipse, Parabola, Hyperbola, General equation of a circle, Equation of tangent and normal at a given point	2
	Intersection of two circles and angle thus generated, Radical axis	2

	Problem	1
	Problem	1
	Equation of a parabola, Equation of tangent and normal	2
	Equation of an ellipse, Equation of tangent and normal	2
	Equation of hyperbola, Equation of tangent and normal	2
	Eccentricity, Rectangular hyperbola Conjugate hyperbola	1
	Doubt Clearing Class	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
	Problem	1
Introduction to 3D	Discussion on 3-dimensional co-ordinates, Distance formula, Division formula	2
	Problem	1

Unit IV	Topic To Be Covered	Number of Classes Required
Limit & Continuity	Idea on limit, Definition of limit i.e. $\lim_{x \to a} f(x) = l$	1
	Some standard formulae, Algebra of limit	1
	Continuity of a function at a point and in an internal, Different types of discontinuity	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
	Problem	1

Derivative	Introduction, Geometrical meaning of derivative, Derivative of a function by ab-initio rule	1
	Derivative of some standard function like $f(x) = c,\ f(x) = e^x, f(x) = a^x\ , f(x) = logx, f(x) = sinx, f(x) = cosx$ etc. Algebra of derivative	1
	Chain rule	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
	Problem	1
Unit V	**Topic To Be Covered**	**Number of Classes Required**
Mathematical Reasoning	Introduction on sentence and statement, Truth value of a statement, Connectives and compound statement	1
	Negation, Conjunction, Disjunction, Implication, Both side implication statements with truth table and suitable examples	1
	Converse, Inverse, Contrapositive statements of a given implication, Equivalent statement, Tautology, Fallacy	1
	Quantifier and predicates	1
Unit V	**Topic To Be Covered**	**Number of Classes Required**
Mathematical Reasoning	Doubt Clearing Class	1
	Problem	1
	Problem	1

Unit VI		
Statistics	Introduction, How to collect data, Sample, Population, grouped and ungrouped data, Arithmetic mean	1
	Geometric mean, Harmonic mean, Median, Mode	1
	Measure of dispersion, Mean deviation, Standard deviation	1
	Variance, Co-efficient of variation	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1
Probability	Introduction, Basic concepts of probability, Sample space, Outcome, Event	1
	Probability of a particular event, Sure event, Impossible event, Fundamental theorems of probability	1
	Axiomatic approach to probability, Finite probability space, Non-uniform space, Mutually exclusive events, Independent events	1
	Conditional probability, Random variable and its use	1
	Doubt Clearing Class	1
	Problem	1
	Problem	1
	Problem	1

LESSON PLAN

(For +2 1st Year)

Overall Revision	**Number of Classes Required**
Unit I	3
Unit II	3
Unit III	3
Unit IV	3
Unit V	1
Unit VI	3

***

MODULE - 1

SET

- A set is a collection of definite distinguished objects of perception or thought.
- Each object belonging to a set is called an element of the set.
- Generally sets are denoted by A, B, C, … etc. and by elements by a, b, …, x, y, … .
- If x is an element of set A we write $x \in$ A otherwise $x \notin$ A.
- Set representation :

  Roster Form : A $= \{1, 2, 3, 4, 5\}$

  Set builder Form : A $= \{x \in N : 1 \leq x \leq 5\}$

- Empty set $\phi = \{\ \}$, Singleton set A $= \{5\}$
- Subset : A is called a subset of B, if each element of A is in B. We denote $A \subseteq B$.
- If $A \subseteq B$ then B is called a super set of A.
- If $A \subseteq B$ and $B \subseteq A$ then $A = B$.
- Some standard sets :-

  $\mathbb{N} =$ Set of natural numbers

  $= \{1, 2, 3, \ldots\}$

  $\mathbb{Z} =$ Set of integers

  $= \{0, \pm 1, \pm 2, \ldots\}$

  $\mathbb{Q} =$ Set of rational numbers

  $S =$ Set of irrational numbers

  $\mathbb{R} =$ Set of real numbers

  $\mathbb{C} =$ Set of complex numbers

- The number of elements in a set A is called its cardinality and denoted by |A|. If |A| is finite we call the set a finite set otherwise infinite set.
- If $|A| = |B|$ then A and B sets are called equivalent.
- Set operation :-

  Union : $A \cup B = \{x : x \in A \text{ or } x \in B\}$

  Intersection : $A \cap B = \{x : x \in A \text{ and } x \in B\}$

  Difference : $A - B = \{x : x \in A,\ x \notin B\}$

  Symmetric Difference : $A \,\Delta\, B = (A \cup B) - (A \cap B)$

- The collection of all subsets of a set A is called its Power set and denoted by $P(A)$.
- Universal Set (E) : It is the super set of all sets under consideration.
- If A is a set then $E - A = A' \; or \; A^c$ is called the complement of set A.
- De' Morgan's Law :

$$(A \cup B)' = A' \cap B'$$

$$(A \cap B)' = A' \cup B'$$

- Generalization of De Morgan's Law

$$\left(\bigcup_{i=1}^{n} A_i\right)' = \bigcap_{i=1}^{n} A_i'$$

$$\left(\bigcap_{i=1}^{n} A_i\right)' = \bigcup_{i=1}^{n} A_i'$$

- Union is distributive over intersection

  i.e. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
- Intersection is distributive over union

  i.e. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Cartesian product :

  We define the Cartesian product of two sets A and B by

$$A \times B = \{(x, y) : \; x \in A, \;\; y \in B\}$$

- In particular

$$A^2 = A \times A = \{(x, y) : \; x, y \in A\}$$

  Similarly,

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y) : \; x, y \in \mathbb{R}\}$$

  Here (x, y) is an ordered pair.

  Again,

$$\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) : \; x_i \in \mathbb{R}, \; \forall \, i\}$$

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 2

RELATION

- A relation R from set A to B is a subset of $A \times B$.
- If $x \in A$ is related to $y \in B$ by relation R, then we write

$$R = \{(x, y) : x \in A,\ y \in B\ s.t.\ xRy\}$$

where xRy read as “x is related to y”

- If $(x, y) \in R$ then the set

$$\{x : x \in A\ and\ for\ some\ y \in B, xRy\}$$

is called the domain of R.

- If $(x, y) \in R$ then the set

$$\{y : y \in B\ and\ for\ some\ x \in A, xRy\}$$

is called the range of R.

- If $R = \{(x, y) : x \in A,\ y \in B\}$

  Then its inverse R^{-1} is defined by

$$R^{-1} = \{(y, x) : xRy\}$$

- Types of Relation :

  One – One Relation : x_1Ry & $x_2Ry \Rightarrow x_1 = x_2$

  Many – One Relation : x_1Ry & $x_2Ry \Rightarrow x_1 \neq x_2$

  One – Many Relation : xRy_1 & $xRy_2 \Rightarrow y_1 \neq y_2$

  Many – Many Relation : If R is One – Many and Many – One

- **Special Types of Relation :**

  Reflexive : A relation R on set A (means A to A) is called reflexive

  If xRx for all $x \in A$

  Symmetric : A relation R on set A is called symmetric

  If $xRx \Rightarrow yRx$

  Transitive : A relation R on set A is called transitive

  If $xRy\ and\ yRz \Rightarrow xRz$

- Equivalence Relation : A relation R on set A is called equivalence if it is reflexive, symmetric and transitive.

- Anti-symmetric Relation : A relation R on set A is called anti-symmetric

  If $xRy\ and\ yRx \Rightarrow x = y$
- Partial Ordering Relation : A relation is called Partial Ordering if it is reflexive, anti-symmetric and transitive.
- Total Ordering Relation : A relation R defined on set A is called total ordering if is partial ordering and for $a, b \in A,\ either\ aRb\ or\ bRa.$
- Relation “Congruence modulo n” on set Z :

  A relation R defined on set Z is called “Congruence modulo n” relation

  If $n \mid a - b$ implies $a \equiv b\ (mod\ n)$ read as “a congruent to b modulo n”
- Congruence modulo n relation on Z is an equivalence relation.
- If R is the equivalence relation on set A then for $a \in A$, the equivalence class of a is defined as $[a] = \{x \in A : aRx\}$.
- An equivalence relation on set A splits the set into disjoint equivalence classes.

Note : Suitable examples and related problems may be incorporated as per necessity.

FUNCTION

- A function f from set A to B is a relation from A to B such that

  (i) It relates all $x \in A$

  (ii) It is either Many – One or One – One.
- For a function $f : A \to B$, Domain of $f = A$
- For a function $f : A \to B$, Co-domain of $f = B$
- Range of $f = \{y \in B : \exists\, x \in A \; which \; comes \; to \; y\}$
- We can write a function $'f'$ from A to B as $f : A \to B$

  Such that $f(x) = y$, where $x \in A, y \in B$
- Here x is the pre-image of y and y is the image of x.
- We may call x as the independent variable and y as the dependent variable.

- **One – One function (Injective function) :**

  A function $f : A \to B$ is called One – One if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$,

  where $x_1, x_2 \in A$.
- **Onto function (Surjective function) :**

  A function $f : A \to B$ is called Onto if Range of f = B.
- **Bijective function :**

  If a function $f : A \to B$ is both One - One and Onto then we call it a bijective function.
- **Identity function :**

  If a function $f : A \to A$ is called an identity function if $f(x) = x, \; \forall\, x \in A$.

Some Standard functions :

- **Constant function :**

  A function $f : A \to B$ is called a constant function

  if $f(x) = c \; (constant) for \; all \; x \in A$.

  Here, Range of $f = c$ (singleton set)
- If $f : X \to Y$ and let A, B are subsets of X then

  (i) $A \subseteq B \Rightarrow f(A) \subseteq f(B)$

  (ii) $f(A \cap B) \subseteq f(A) \cap f(B)$

  (iii) $f(A \cup B) = f(A) \cup f(B)$

- **Real function (Real valued function) :**

  A function $f : A \to B$ is called a real function if both A and B are subsets of $\mathbb{R}$.

- Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and let $D = A \cap B$,

  then on the domain D we define

$$(f + g)(x) = f(x) + g(x)$$
$$(f - g)(x) = f(x) - g(x)$$
$$(f.g)(x) = f(x).g(x)$$
$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}, g(x) \neq 0$$

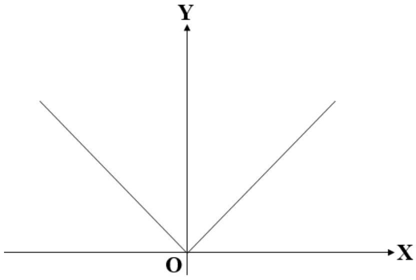
- **Odd function :** A function f is called odd if $f(-x) = -f(x)$
- **Even function :** A function f is called even if $f(-x) = f(x)$

Some useful functions :

- **Absolute value function :** A function $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = |x|$, where

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

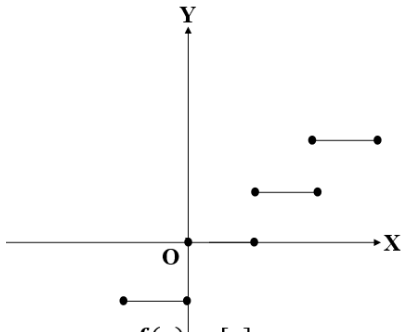
  is called absolute value or modulus function.



$f(x) = |x|$

- **Bracket function (Greatest Integer function) :**

  A function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = [x]$, where [x] is the greatest integer less than or equal to x.

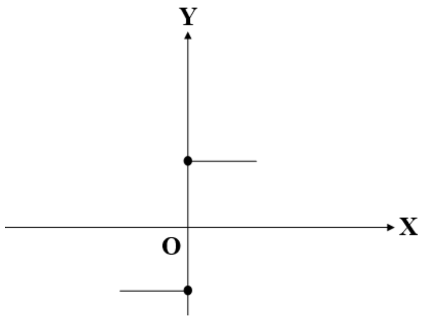


$f(x) = [x]$

- **Signum function :** A function $f : \mathbb{R} \to \mathbb{R}$ defined by

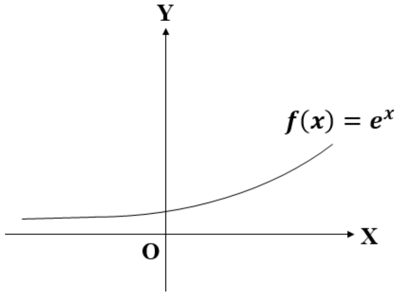
$$sgn(x) = \begin{cases} \frac{|x|}{x}, & if\ x \neq 0 \\ 0, & if\ x = 0 \end{cases}$$

  is called Signum function.

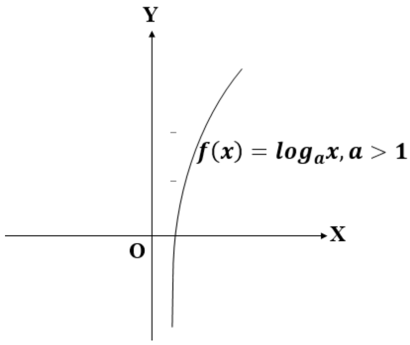


$f(x) = sgn(x)$

- **Exponential function :** A function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = a^x, a \in \mathbb{R}^+$ is called exponential function.
  Special case, $f(x) = e^x$



- **Logarithmic function :** A function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \log_a x$,
  where $a \in \mathbb{R}^+$ is called logarithmic function.

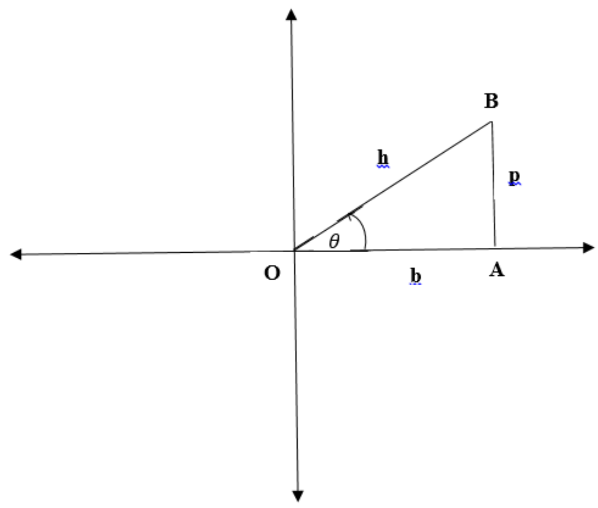


- **Polynomial function :** A function defined by $f : \mathbb{R} \to \mathbb{R}$ such that
  $f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, where $a_0, a_1, \ldots, a_n \in \mathbb{R}$ and not all zero, is called a polynomial function.
- **Rational function :** A function $f : \mathbb{R} \to \mathbb{R}$ defined by
  $$f(x) = \frac{P(x)}{Q(x)}$$
  where P(x) and Q(x) are polynomials, is called rational function.
  It is defined $\forall\ x\ such\ that\ Q(x) \neq 0$

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 3

TRIGONOMETRIC FUNCTIONS



- In $\Delta\, ABO$ we define

  $\sin\theta = \frac{p}{h}$, $\cos\theta = \frac{b}{h}$

  $\tan\theta = \frac{p}{b}$, $\cot\theta = \frac{b}{p}$

  $\sec\theta = \frac{h}{b}$, $\text{cosec}\,\theta = \frac{h}{p}$

- The angle θ is + ve when measured in anti-clockwise direction and – ve when measured in clockwise direction.

- **System of Measurement :**

  Sexagesimal (Degree) : 1 right angle = 90^0

  $1^0 = 60'$ (60 minute)

  $1' = 60''$ (60 second)

  Centesimal (Grade) : 1 right angle = 100^g

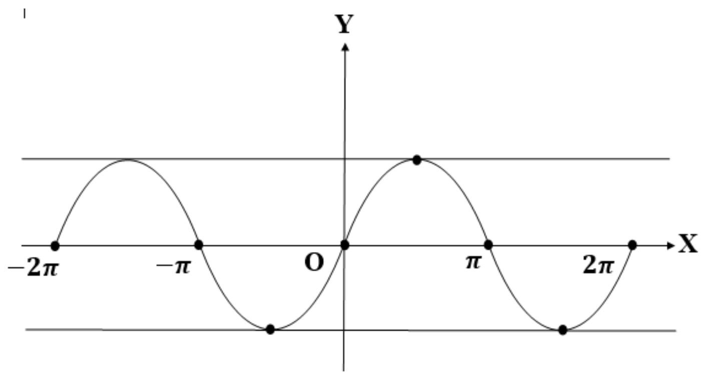
  $1^g = 100'$

  $1' = 100''$

  Circular (Radian): 1 right angle = $\frac{\pi}{2}^c$

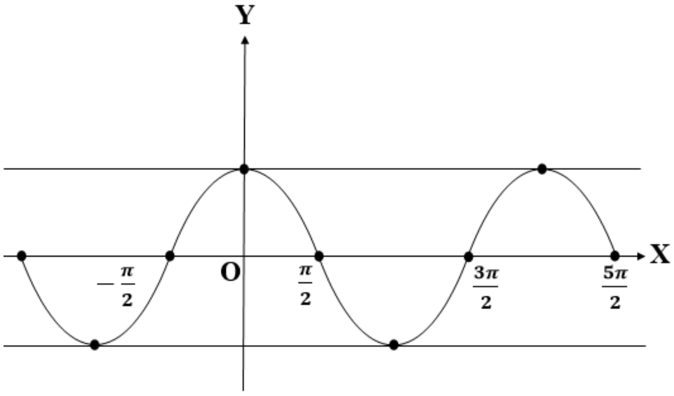
  So, $200^g = 180^0 = \pi^c = 2$ right angles

- Figure of $f(x) = \sin x$



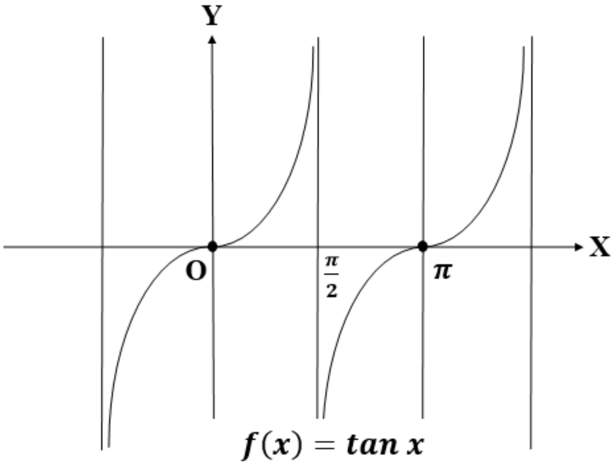
$f(x) = \sin x$

Figure of $f(x) = \cos x$



$f(x) = \cos x$

Figure of $f(x) = \tan x$



$f(x) = \tan x$

- Domain of $f(x) = \sin x$ is $\mathbb{R}$ and Range is $[-1, 1]$
- Domain of $f(x) = \cos x$ is $\mathbb{R}$ and Range is $[-1, 1]$
- Domain of $f(x) = \tan x$ is $\mathbb{R}$ and Range is $(-\infty, \infty)$
- **Some general identities :**

  (i) $\sin^2\theta + \cos^2\theta = 1$

  (ii) $\sec^2\theta = 1 + \tan^2\theta$

  (iii) $\text{cosec}^2\theta = 1 + \cot^2\theta$

- Trigonometric ratios of the angles associated with $90^0, 180^0, 270^0, \ldots$

  (i) $\sin(90^0 - \theta) = \cos\theta, \cos(90^0 - \theta) = \sin\theta$

  $\tan(90^0 - \theta) = cot\theta,\ \cot(90^0 - \theta) = \tan\theta$ and so on.

  (ii) $\sin(90^0 + \theta) = \cos\theta,\ \cos(90^0 + \theta) = -\sin\theta$ and so on.

  (iii) $\sin(180^0 - \theta) = \sin\theta, \cos(180^0 - \theta) = -\cos\theta$ and so on.

  (iv) $\sin(180^0 + \theta) = -\sin\theta, \cos(180^0 + \theta) = -\cos\theta$ and so on.

Other formulae can be derived in this manner.

- **Periodicity of a function :**

  If $f(x+k) = f(x)\ \forall\ x$ in the domain, then we call it a periodic function with period $k\ (> 0)$.

- Period of $f(x) = \sin x$ is 2π

  Period of $f(x) = \tan x$ is π

- **Trigonometric ratios for compound & multiple angles :**

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$tan\ (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$tan\ (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$\sin A + \sin B = 2 \sin\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\sin A - \sin B = 2 \cos\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\cos A + \cos B = 2 \cos\left(\frac{A+B}{2}\right) \cos\left(\frac{A-B}{2}\right)$$

$$\cos A - \cos B = -2\ \sin\left(\frac{A+B}{2}\right) \sin\left(\frac{A-B}{2}\right)$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A}$$

$$\sin 3A = 3 \sin A - 4 \sin^3 A$$

$$\cos 3A = 4 \cos^3 A - 3 \cos A$$

$$\tan 3A = \frac{3 \tan A - \tan^3 A}{1 - 3 \tan^2 A}$$

- **Trigonometric equations :**

  Equations like $\sin x = \frac{1}{2}, \cos \text{x} = 1, \tan 2x + \tan x = 1, etc.$ are called trigonometric equations.

- Suppose the given trigonometric equation is $\sin x = \frac{1}{2}$,

  then its principal solution is $x = 30^0, 150^0$ (solution lies between 0^0 and 360^0)

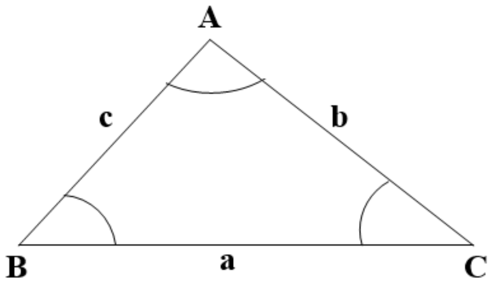
  and its general solutions are

$$x = 2n\pi + \frac{\pi}{6}\ , n \in \mathbb{Z}\ and\ x = 2n\pi + \frac{5\pi}{6}\ , n \in \mathbb{Z}$$

- **Equation and its general solution :**

$$sin\ x = 0,\ \text{general solution } x = n\pi,\ n \in \mathbb{Z}$$
$$cos\ x = 0,\ \text{general solution } x = (2n+1)\frac{\pi}{2}\ ,\ n \in \mathbb{Z}$$
$$tan\ x = 0,\ \text{general solution } x = n\pi,\ n \in \mathbb{Z}$$
$$sin\ x = \sin\alpha,\ \text{general solution } x = n\pi + (-1)^n\alpha,\ n \in \mathbb{Z}$$
$$cos\ x = \cos\alpha,\ \text{general solution } x = 2n\pi \pm \alpha,\ n \in \mathbb{Z}$$
$$tan\ x = \tan\alpha,\ \text{general solution } x = n\pi + \alpha,\ n \in \mathbb{Z}$$

Properties of Triangles :



In any $\Delta\ ABC$

- Sine formula :

$$\frac{a}{sin\ A} = \frac{b}{sin\ B} = \frac{c}{sin\ C}$$

- Cosine formula :

$$a^2 = b^2 + c^2 - 2bc\ cos\ A$$
$$b^2 = a^2 + c^2 - 2ac\ cos\ B$$
$$c^2 = a^2 + b^2 - 2ab\ cos\ C$$

- Projection formula :

$$a = b\cos C + c\cos B$$
$$b = a\cos C + c\cos A$$
$$c = a\cos B + b\cos A$$

- **Napier's formula (Tangent formula) :**

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot\frac{A}{2}$$
$$\tan\left(\frac{C-A}{2}\right) = \frac{c-a}{c+a}\cot\frac{B}{2}$$
$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

- Semi-angle formula :

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}, \qquad \cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

where $2s = a + b + c$

- Area formula :

$$\text{Area} = \Delta = \sqrt{s(s-a)(s-b)(s-c)} \text{ (Heron's Formula)}$$

and also we have

$$\Delta = \frac{1}{2}\ ab \sin C = \frac{1}{2}\ ca \sin B = \frac{1}{2}\ bc \sin A$$

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 4

COMPLEX NUMBERS

- The number $a+ib$, where $a, b \in \mathbb{R}$ and i is a symbol used for $\sqrt{-1}$, is called a complex number. The set of all complex numbers is denoted by $\mathbb{C}$.
- Equality, if $a+ib=c+id$, then $a=c, b=d$.
- Addition : $(a+ib)+(c+id)=(a+c)+i(b+d)$
- Multiplication : $(a+ib)(c+id)=(ac-bd)+i(ad+bc)$
- Integral power of i is given by

$$i^n = \begin{cases} 1 & if\ n=4k \\ i & if\ n=4k+1 \\ -1 & if\ n=4k+2 \\ -i & if\ n=4k+3 \end{cases}$$

  where k is an integer.
- For $z=a+ib$, the real and imaginary parts are $Rl(z)=a,\ Im\ (z)=b$ respectively.
- If $z=x+iy$ then its conjugate is defined as $\bar{z}=x-iy$.
- If $z=x+iy$ then its modulus is defined by $|z|=\sqrt{x^2+y^2}$
- Division of two complex numbers :

$$\frac{z_1}{z_2}=\frac{z_1\ \overline{z_2}}{z_2\ \overline{z_2}}=\frac{z_1\ \overline{z_2}}{|z_2|^2}\ ,(z_2\neq 0)$$

- If $z=x+iy$ then its inverse is given by

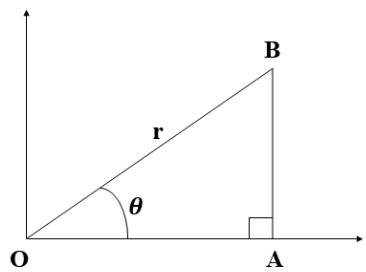
$$z^{-1}=\frac{1}{z}=\frac{\bar{z}}{|z|^2}$$

- **Polar form :**

  The complex number $z=x+iy$ may be written as

$$z=r(\cos\theta+i\ \sin\theta)$$

  Here $|z|=r$ and θ is called the argument of z, $(-\pi\leq\theta\leq\pi)$



- **Square root of a complex number :**

  Let $x+iy$ be the square root of $a+ib$. Then

$$a+ib=(x+iy)^2$$
$$=x^2-y^2+i\ 2xy$$

So, $a=x^2-y^2, b=2xy$ and from this we can solve for x, y.

- **Cube root of unity :**

  If we consider w as the cube root of 1, then

$$w^3 = 1$$
$$\Rightarrow w^3 - 1 = 0$$
$$\Rightarrow (w-1)(w^2 + w + 1) = 0$$

  which gives $w = 1, w = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

- **De-Moivre's Theorem :**

  For integral index :

$$(cos\ \theta + i\ sin\ \theta)^n = \cos n\theta + i \sin n\theta\,,\ \ n \in \mathbb{Z}$$

  For rational index :

  Let $p, q \in \mathbb{Z}$ with $q > 0$, then

$$\left(\cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta\right)^q = \cos p\theta + i\sin p\theta$$
$$= (\cos\theta + i\sin\theta)^p$$

  and this implies

$$(\cos\theta + i\sin\theta)^{\frac{p}{q}} = \cos\frac{p}{q}\theta + i\sin\frac{p}{q}\theta$$

- If $z = r(\cos\theta + i\sin\theta)$ is a complex number then its $\boldsymbol{nth}$ roots are

$$z^{\frac{1}{n}} = r^{\frac{1}{n}}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

  where $k = 0, 1, 2, \ldots, n-1$

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 5

LINEAR INEQUALITIES

- A statement involving variables and the sign of inequality $>, <, \geq, \leq$ is called an inequality or inequation.

  e.g. $ax + b < 0$

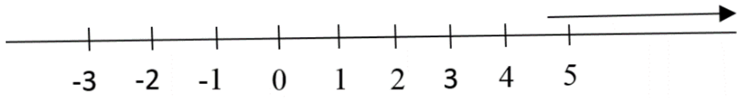
  $ax + b \leq 0$

  $ax + by > 0$

  $ax + by \geq 0$, etc.

- Solve $x + 12 < 4x - 2$

  Solution : $x + 12 < 4x - 2$

  $\Rightarrow 3x > 14$

  $\Rightarrow x > \frac{14}{3}$

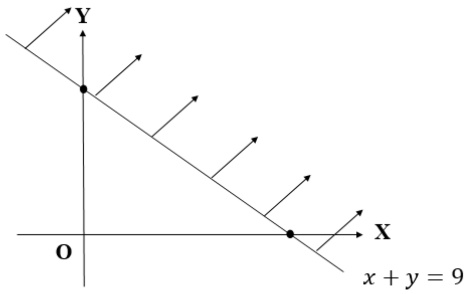
- **Graphical solution of linear inequality (with two variables) :**

  Solve graphically $y + x + 5 > 14$

  Solution : Here $x + y > 9$

  $x + y = 9$

x	0	9
y	9	0



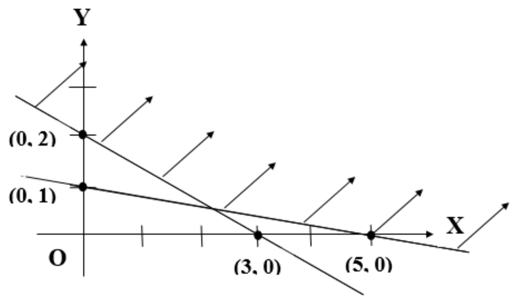
Shaded region is the necessary graphical solution.

- **Graphical solution of system of inequalities :**

  Solve graphically –

  $$2x + 3y \geq 6$$

  $$x + 5y \geq 5$$



Solution :

$2x + 3y = 6$

x	0	3
y	2	0

$x + 5y = 5$

x	0	5
y	1	0

Shaded region is the necessary graphical solution.

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 6

PERMUTATION & COMBINATION

- **Principle of Counting (Product Rule) :**

  If an event occurs in p different ways and following which another event can occur in q different ways, then the total number of ways for the occurrence of both the events = $p \times q$

- **Principle of Counting (Sum Rule) :**

  If one event can occur in either p ways or q ways of separate categories then the number of ways of occurrence = $p + q$

  Examples : Product Rule –

  If there are 4 ways to cover the distance from BBSR to CTC and then we have 3 ways to cover distance from CTC to DKL, then the total number of ways from BBSR to DKL = $4 \times 3 = 12$.

  Examples : Sum Rule –

  If a student has to choose one game from 3 indoor & 5 outdoor categories then the number of choices = $3 + 5 = 8$.

PERMUTATION

- It is an arrangement of objects taking some or all at a time. (Here ordering is important)
- If we have n distinct objects then the number of permutations taking r at a time $(r \leq n)$

$$p(n, r) = \frac{n!}{(n-r)!}$$

- The number of permutations of n distinct objects taking r at a time where repetition is allowed is given by

$$\underbrace{n \times n \times \dots \times n}_{r-times} = n^r$$

- The number of permutations of n objects where p number are of one kind, q number are of second kind and r number are of third kind is given by

$$\frac{n!}{p!\,q!\,r!}$$

- **Circular Permutation :**

  The number of circular permutations of n objects is $(n-1)!$

COMBINATION

- An arrangement of n objects taking some or all at a time where the ordering in each arrangement is meaningless is called a combination.
  e.g. if we have 5 players and a team of 2 to be selected, then each choice is called a combination.
- The number of combinations of n objects taking r at a time (r ≤ n) is given by
  $$C(n,r) = {}^{n}C_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$
- **Complementary Combination :**
  $$C(n,r) = C(n, n-r)$$
  are called complementary combinations.
- $C(n,r) + C(n,r-1) = C(n+1,r)$
- $r\,C(n,r) = n\,C(n-1,r-1)$
- $\frac{C(n,\ r)}{C(n,\ r-1)} = \frac{n-r+1}{r}$
- ${}^{n}C_0 + {}^{n}C_1 + \cdots + {}^{n}C_n = 2^n$
- The number of ways in which $p+q$ different things are divided into two groups of p and q things is $\frac{(p+q)!}{p!q!}$
- If $p = q$, then the groups are of equal strength and number of combinations = $\frac{(2p)!}{2!p!q!}$
- If $2p$ things are distributed equally between 2 persons, then number of ways = $\frac{(2p)!}{p!p!}$

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 7

BINOMIAL THEOREM

- When n is a $+ve$ integer

$$(a+b)^n = {}^nC_0 a^n b^0 + {}^nC_1 a^{n-1} b^1 + \cdots + {}^nC_n a^0 b^n$$

  This is called as Binomial Theorem.
- If the index is n, then the number of terms is $n+1$.
- If we take $a = 1, b = x$, then

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$$
$$= C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$$

- **General Term :**

  For the binomial expansion $(a+b)^n$ we get $T_{r+1} = term$

  number $(r+1) = {}^nC_r a^{n-r} b^r$

Middle Term (s) :

- If for $(a+b)^n$, the number n is even, then the middle term is $T_{\frac{n}{2}+1} = C\left(n, \frac{n}{2}\right) a^{n-\frac{n}{2}} b^{\frac{n}{2}}$
- If for $(a+b)^n$, the number n is odd, then there are two middle terms $T_{\frac{n+1}{2}}, T_{\frac{n+1}{2}+1}$

Equidistant Terms :

- In the expansion of $(a+b)^n$, the coefficients of $(r+1)$th term from the beginning and $(r+1)$th term from the end are equal.
- The number of terms in the expansion $(a+b+c)^n$ is $\frac{(n+1)(n+2)}{2}$,

  where n is +ve integer.
- The number or terms in the expansion of $(a+b+c+d)^n$ is $\frac{(n+1)(n+2)(n+3)}{1.2.3}$
- For binomial coefficients we have

$$C_0 + C_2 + C_4 + \cdots = C_1 + C_3 + C_5 + \cdots = 2^{n-1}$$

- **Some Special Cases :**

  (i) $\frac{1}{1-x} = (1-x)^{-1} = 1 + x + x^2 + \cdots \quad |x| < 1$

  (ii) $\frac{1}{1+x} = (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots \quad |x| < 1$

  (iii) $\frac{1}{(1+x)^2} = (1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \cdots \quad |x| < 1$

  (iv) $\frac{1}{(1-x)^2} = (1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \cdots \quad |x| < 1$

- **General Expansion :**

  $(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \cdots$ where $|x| < 1$ and α is any rational number.

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 8

SEQUENCE AND SERIES

- A set of numbers arranged in a definite order according to some rule is called a sequence.

  e.g. 1, $\frac{1}{2}$, $\frac{1}{2^2}$, $\frac{1}{2^3}$, ...

  Here the general term $a_n = \frac{1}{2^{n-1}}, n \in N$

  Generally the terms of a sequence are written as $a_1, a_2, \dots, a_n, \dots$

- **Finite Sequence :**

$$\{2, 4, 6, \dots\ 2n\}$$

- **Infinite Sequence :**

$$\{1, 2, 3, 4, \dots\}$$

- **Fibonacci Sequence :**

  This is defined by 1, 1, 2, 3, 5, …

  Here $a_1 = 1, a_2 = 1, \dots$, then $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$

- **Series :**

  If $a_1, a_2, \dots, a_n, \dots$ is a sequence, then $a_1 + a_2 + \cdots + a_n + \cdots$

$$= \sum_{i=1}^{\infty} a_i$$

  is called an infinite series.

- The partial sum S_n of a series is given by

$$S_n = a_1 + a_2 + \cdots + a_n$$
$$= \sum_{i=1}^{n} a_i$$

- **Arithmetic Series :**

  If $t_{n+1} - t_n = d$ (constant) for $n = 1, 2, 3, ..$ then (t_n) is called arithmetic sequence or Arithmetic Progression (A.P.) and the series $\sum t_n$ is called an Arithmetic Series.

- **Partial Sum of Arithmetic Series :**

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

  where $t_1 = a$, Common difference = d

- **Geometric Series :**

  If $\frac{t_{r+1}}{t_r} = a$ (constant) for $n = 1, 2, 3, \ldots$ then (t_n) is called a geometric sequence or Geometric Progression (G.P.) and the series $\sum t_n$ is called Geometric Series.

- The nth partial sum of geometric series is given by

$$S_n = \frac{a(1 - r^n)}{1 - r} \; for \; |r| < 1$$

$$Also \qquad S_n = \frac{a(r^n - 1)}{r - 1} \; for \; |r| > 1$$

  where $a = t_1$ and r is the common ratio.

  For $r = 1$, $S_n = a + a + \cdots \; to \; n \; terms = na$

- **Arithmetico-geometric Series :**

  If (a_n) is AP and (b_n) is GP then $(a_n b_n)$ is called Arithmetico-geometric Sequence and $\sum a_n b_n$ is called Arithmetico-geometric Series.

- The arithmetic series always diverges except $t_1 = a = 0$ and $d = 0$

- **Sum of Geometric Series :**

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1 - r} \; if \; |r| < 1$$

$$= \infty \quad if \; |r| \geq 1$$

- **Sum of Arithmetico-geometric Series :**

  For $|r| < 1$ the Arithmetico-geometric series converges, $\lim_{n \to \infty} S_n = \frac{ab}{1-r} + \frac{dbr}{(1-r)^2}$

  and we have $ab + (a + d)br + (a + 2d)br^2 + \cdots = b\left[\frac{a}{1-r} + \frac{dr}{(1-r)^2}\right]$

  Some Special Series Expansion :

- $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots, \; x \in R$

- For $a > 0, y \in R$

$$a^y = 1 + y log_e a + \frac{y^2 (\log_e a)^2}{2!} + \cdots$$

Note : Suitable examples and related problems may be incorporated as per necessity.

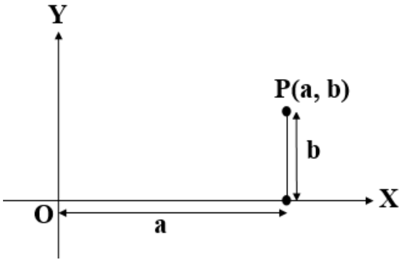
MODULE - 9

2- DIMENSIONAL CO-ORDINATE GEOMETRY

STRAIGHT LINE

- Coordinate geometry explains geometrical figures in terms of algebraic equation.
- It is divided into two parts,

  (a) 2D (b) 3D
- 2- dimensional coordinate geometry also known as plane coordinate geometry.



- On a plane we need a Cartesian coordinate axis system to locate a point P.

  Generally we identify P by coordinates (a, b).

  Here a is the X- coordinate (abscissa),

  b is the Y- coordinate or ordinate.

- **Distance Formula :**

  If $P(x_1, y_1)$ and $Q(x_2, y_2)$ are two distinct points in a plane then the distance between them is $|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

  $$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

  Division Formula :

- If $R(x, y)$ divides the line joining $P(x_1, y_1)$ and $Q(x_2, y_2)$ in $m : n$ ratio internally then

$$x = \frac{mx_2 + nx_1}{m + n}, \qquad y = \frac{my_2 + ny_1}{m + n}$$

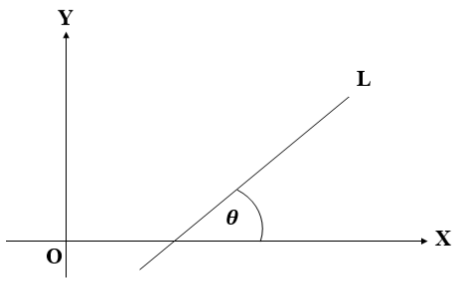
- If $R(x, y)$ is the midpoint of $\overline{PQ}$ then

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}$$

- If $R(x, y)$ divides $\overline{PQ}$ externally in $m : n$ ratio then

$$x = \frac{mx_2 - nx_1}{m - n}, \ y = \frac{my_2 - ny_1}{m - n}$$

  Slope of a line :



- If a line L makes an angle θ with the +ve direction of X- axis then its slope is defined as $tan\theta = m\ (say)$
- The Y- axis or a line parallel to Y- axis has no slope.

- If L_1 and L_2 are the two lines with slope m_1 and m_2 respectively then the angle between them is given by

$$\theta = \tan^{-1}\left[\pm\frac{m_1 - m_2}{1 + m_1 m_2}\right]$$

- **Locus :**

  The path trace out by a point which moves obeying certain geometrical condition is called a locus.

  e.g. Straight line, Circle, Parabola, etc.

- **Straight Line and its Equation :**

  If a straight line has slope m and Y- intercept c then its equation will be $y = mx + c$

- **Point – Slope form equation :**

  If a straight line passes through (x_1, y_1) and its slope is m then the equation will be

$$y - y_1 = m(x - x_1)$$

- **2- Point Form :**

  If a straight line passes through 2 points (x_1, y_1) and (x_2, y_2) then its equation will be

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$$

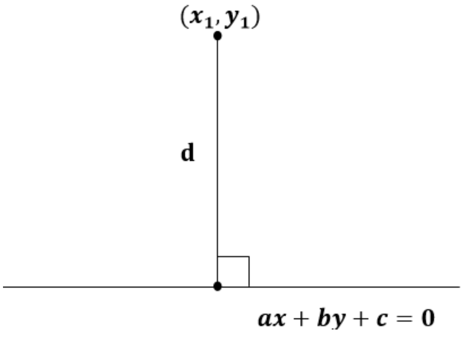
- General equation of a straight line is $ax + by + c = 0$
- If $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ are two straight lines then their point of intersection is

$$\left(\frac{b_1c_2 - c_1b_2}{a_1b_2 - a_2b_1}, \frac{c_1a_2 - a_1c_2}{a_1b_2 - a_2b_1}\right)$$

- When $a_1x + b_1y + c_1 = 0$ is parallel to $a_2x + b_2y + c_2 = 0$ we find

$$\frac{a_1}{a_2} = \frac{b_1}{b_2}$$

- If two lines are perpendicular then $a_1a_2 + b_1b_2 = 0$
- The perpendicular distance of a point $P(x_1, y_1)$ from a line $ax + by + c = 0$ is



$$d = \left|\frac{ax_1 + by_1 + c}{\sqrt{a^2 + b^2}}\right|$$

- Pair of straight lines is represented by the equation $ax^2 + 2hxy + by^2 = 0$
- The angle between the pair of straight lines ($ax^2 + 2hxy + by^2 = 0$) is

$$\theta = \tan^{-1}\left[\pm 2\frac{\sqrt{h^2 - ab}}{a + b}\right], \qquad h^2 \geq ab$$

- If $ax^2 + 2hxy + by^2 = 0$ is a pair of straight lines then its bisectors are given by

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h} \quad if \;\; a \neq b$$

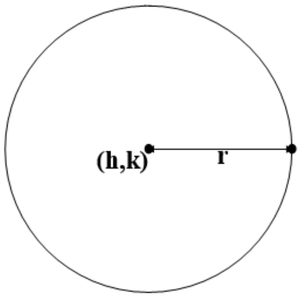
Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 10

CONIC SECTIONS :

CIRCLE

- The locus of a point which moves at a fixed distance from a fixed point is called a circle. The fixed point is the centre and the fixed distance is the radius.
- If (h, k) is the centre and r is the radius of a circle then its equation will be
$$(x-h)^2 + (y-k)^2 = r^2$$
- If $A(x_1, y_1)$ and $B(x_2, y_2)$ are end points of the diameter of a circle then its equation will be $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$
- General equation of a circle is written as $x^2 + y^2 + 2gx + 2fy + c = 0$
- If a circle has centre $(0, 0)$ and radius r then its equation will be $x^2 + y^2 = r^2$
- If $S_1 = x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and $S_1 = x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ are two circle intersect each other at two points then all the circles pass through those two points are represented by
$$S_1 + kS_2 = 0, \qquad k \in R, \qquad k \neq -1$$

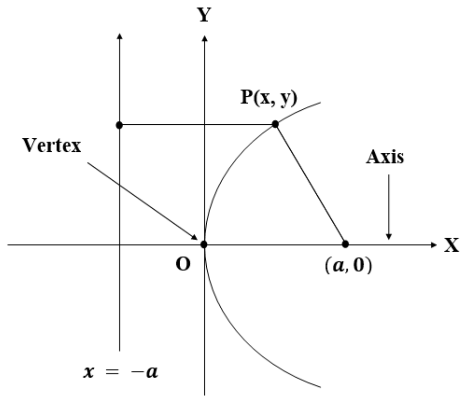


- If $k = -1$ then
$$S_1 + kS_2 = 0$$
$$\Rightarrow S_1 - S_2 = 0$$
$$\Rightarrow 2x(g_1 - g_2) + 2y(f_1 - f_2) + (c_1 - c_2) = 0$$
which is a straight line and called as **radial axis**.
- Equation of the tangent to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is
$$xx_1 + yy_1 + g(x + x_1) + f(y + y_1) + c = 0$$
- Equation of the normal to a circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at (x_1, y_1) is
$$y - y_1 = \frac{y_1 + f}{x_1 + g}(x - x_1)$$
- A line $y = mx + c$ becomes a tangent to a circle $x^2 + y^2 = a^2$ if $c = \pm a\sqrt{1 + m^2}$

Note : Suitable examples and related problems may be incorporated as per necessity.

PARABOLA

- A set of points equidistant from a fixed point and a fixed line is called the parabola.
- The fixed point is called the focus and the fixed line is the directrix.
- The line passes through the focus and perpendicular to the directrix is called the axis of the parabola.
- The line passes through the focus and perpendicular to the axis is called latus rectum.



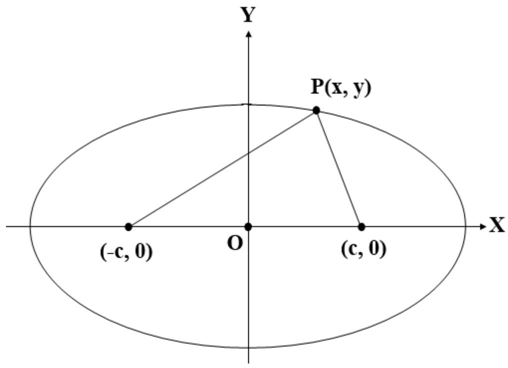
- The equation of the parabola with focus $(a, 0)$, vertex $(0, 0)$ and directrix $x = -a$ is given by $y^2 = 4ax$ (here the axis is X- axis)
- If vertex is $(0, 0)$, focus is $(0, a)$ then the axis becomes Y- axis and the equation of the parabola will be $x^2 = 4ay$ (here the directrix is $y = -a$)
- Equation of a parabola with vertex (h, k), focus $(h + a, k)$ and axis parallel to X- axis is given by $(y - k)^2 = 4a(x - h)$
- Equation of a parabola with vertex (h, k), focus $(h, k + a)$ and axis parallel to Y- axis is given by $(x - h)^2 = 4a(y - k)$
- The equation of the tangent to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is
$$yy_1 = 2a(x + x_1)$$
- The equation of the normal to the parabola $y^2 = 4ax$ at a point (x_1, y_1) is
$$y - y_1 = -\frac{y_1}{2a}(x - x_1)$$

- **Condition of tangency :**

  A line $y = mx + c$ becomes tangent to a parabola $y^2 = 4ax$ if $c = \frac{a}{m}$
- Parametric equation of a parabola $y^2 = 4ax$ is given by $x = at^2, y = 2at$

Note : Suitable examples and related problems may be incorporated as per necessity.

ELLIPSE



- An ellipse is defined as the set of points on a plane such that the sum of its distances from two fixed points on a plane is constant.
- Two fixed points are called foci.
- The line passes through the foci is called the major axis.
- The points where the ellipse cuts its major axis are called vertices.
- The lines pass through the foci and perpendicular to the axis are called latera recta (plural of latus rectum).
- The midpoint of the line joining the foci is called the centre.
- The line through centre and perpendicular to the major axis is called minor axis.
- The equation of the ellipse with centre $(0, 0)$ and foci $(\pm c, 0)$ and vertices $(\pm a, 0)$ is given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad where\ b^2 = a^2 - c^2$$

- The equation of the ellipse with centre $(0, 0)$, foci $(0, \pm c)$ and vertices $(0, \pm a)$ is given by

$$\frac{y^2}{a^2} + \frac{x^2}{b^2} = 1, \qquad where\ b^2 = a^2 - c^2$$

- If the ellipse has (h, k) centre, $(h \pm c, k)$ foci and $(h \pm a, k)$ as vertices then its equation will be

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

- If the ellipse has (h, k) centre, $(h, k \pm c)$ foci and $(h, k \pm a)$ as vertices then its equation will be

$$\frac{(y-k)^2}{a^2} + \frac{(x-h)^2}{b^2} = 1$$

- For an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the tangent at (x_1, y_1) is given by

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

- For an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ the normal at (x_1, y_1) is given by

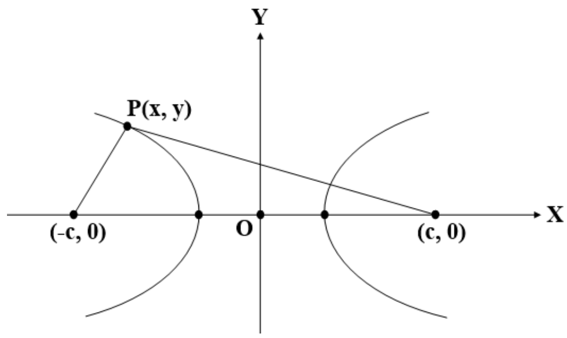
$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

- A line $y = mx + c$ will be a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ if $c^2 = a^2 m^2 + b^2$
- Parametric form equation of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is $x = a\ cos\theta,\ y = b\ sin\theta$
- If $a = b$ then the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ becomes a circle.

Note : Suitable examples and related problems may be incorporated as per necessity.

HYPERBOLA

- A hyperbola is the locus of a point in the plane such that the magnitude of the difference of its distances from two fixed points is constant.
- Fixed points are called foci.
- The line through foci is called the major axis.
- The points where the hyperbola cuts its major axis are the vertices.



- The lines through foci and perpendicular to the major axis are called latera recta (plural of Latus rectum)
- Suppose a hyperbola has foci $(\pm c, 0)$, vertices $(\pm a, 0)$ then its equation will be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad where\ b^2 = c^2 - a^2$$

- Suppose a hyperbola has foci $(0, \pm c)$, vertices $(0, \pm a)$ then its equation will be

$$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1, \qquad where\ b^2 = c^2 - a^2$$

- If the hyperbola has $(h \pm c\ , k)$ foci and $(h \pm a, k)$ as vertices then its equation will be

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

- If the hyperbola has $(h, k \pm c)$ foci and $(h, k \pm a)$ as vertices then its equation will be

$$\frac{(y-k)^2}{a^2} - \frac{(x-h)^2}{b^2} = 1$$

- For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the tangent at (x_1, y_1) is given by

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$$

- For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ the normal at (x_1, y_1) is given by

$$y - y_1 = -\frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

- A line $y = mx + c$ will be a tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ if $c^2 = a^2m^2 - b^2$
- Parametric form equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $x = a\ sec\theta,\ y = b\ tan\theta$
- Parametric form equation of the hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ is $x = b\ cot\theta,\ y = a\ cosec\theta$

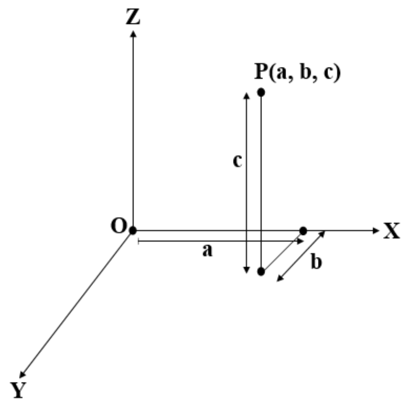
- If $a = b$ then the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is called **rectangular hyperbola**.
- The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ are conjugate to each other.

Special Feature :

- Eccentricity of a conic section is $e = \frac{PF}{PD}$, where PF is the distance of any point on the conic from focus and PD is its distance from the directrix.
- For parabola $e = 1$
- For ellipse $e < 1$
- For hyperbola $e > 1$

Note : Suitable examples and related problems may be incorporated as per necessity.

3- DIMENSIONAL CO-ORDINATE GEOMETRY



- In space we can identify a point P with 3 coordinates $(a,\ b,\ c)$
- We use an axis system consists of 3 mutually perpendicular lines X- axis, Y- axis, Z- axis to locate any point in space.
- The distance between two points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is

$$|PQ| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$$

- The point $R(x, y, z)$ which divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in ratio $m : n$ is given by

$$x = \frac{mx_2 + nx_1}{m + n}, \qquad y = \frac{my_2 + ny_1}{m + n}, \qquad z = \frac{mz_2 + nz_1}{m + n}$$

This is called as Division Formula.

- The midpoint of $\overline{PQ}$ is

$$\left(\frac{x_1 + x_2}{2}, \qquad \frac{y_1 + y_2}{2}, \qquad \frac{z_1 + z_2}{2}\right)$$

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 11

LIMIT

- The limit of a function $f(x)$ $at\ a\ point$ $x = a$ is equal to $'l'$ if for a small $\epsilon > 0$ there exists $\delta > 0$ (depending on ϵ) such that

  $|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$

  We write $\lim_{x \to a} f(x) = l$

- **Left Hand Limit :**

  If $|f(x) - l| < \epsilon$ when $-\delta < x < a$,

  then we call this as left hand limit at 'a', i.e. $\lim_{x \to a^-} f(x) = l$

- **Right Hand Limit :**

  If $|f(x) - l| < \epsilon$ when $< x < a + \delta$,

  then we call this as left hand limit at 'a', i.e.

  $\lim_{x \to a^+} f(x) = l$

- If $\lim_{x \to a^+} f(x) = \lim_{x \to a^-} f(x) = l$,

  then only $\lim_{x \to a} f(x) = l$ (limit exists)

- **Algebra of Limit :**

  If $\lim_{x \to a} f(x) = l$, $\lim_{x \to a} g(x) = m$

  Then (i) $\lim_{x \to a}[f(x) \pm g(x)] = l \pm m$

  (ii) $\lim_{x \to a}[f(x). g(x)] = l.m$

  (iii) $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{l}{m}$ $(m \neq 0)$

- If $|f(x) - l| < \epsilon$ for $x > M$ where M is very large $+ve$ number, we write

  $\lim_{x \to \infty} f(x) = l$

- If $|f(x) - l| < \epsilon$ for $x < -M$ where M is very large $+ve$ number, we write

  $\lim_{x \to -\infty} f(x) = l$

- If for $|x - a| < \delta$ we find $f(x) > M$ (M is very large $+ve$ number) then

  $\lim_{x \to a} f(x) = \infty$

- If for $|x - a| < \delta$ we find $f(x) < -M$ (M is very large $+ve$ number) then

  $\lim_{x \to a} f(x) = -\infty$

Some Standard Formulae :

- $\lim\limits_{x \to a} \frac{x^n - a^n}{x - a} = na^{n-1}$ (n is rational)
- $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$
- $\lim\limits_{x \to 0} \frac{\tan x}{x} = 1$
- $\lim\limits_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e$
- $\lim\limits_{x \to 0} \frac{\log_e(1+x)}{x} = 1$
- $\lim\limits_{x \to 0} \frac{a^x - 1}{x} = \log_e a \ \ (a > 0)$
- $\lim\limits_{x \to 0} \frac{\sin ax}{x} = a \lim\limits_{x \to 0} \frac{\sin x}{ax} = a$
- For a function $f(x)$,

  $$\lim_{x \to a} kf(x) = k \ \lim_{x \to a} f(x)$$

  where k is a constant.
- $\lim\limits_{x \to a} \frac{e^x - 1}{x} = 1$
- $\lim\limits_{x \to a} [f(x)]^n = \left[\lim\limits_{x \to a} f(x)\right]^n$
- $\lim\limits_{x \to a} fog(x) = f\left[\lim\limits_{x \to a} g(x)\right]$

Note : Suitable examples and related problems may be incorporated as per necessity.

<u>CONTINUITY</u>

- A function $f(x)$ is said to be continuous at a point $x = a$ if

$$\lim_{x \to a} f(x) = f(a)$$

- A function $f(x)$ is said to be continuous from the left at $x = a$ if

$$\lim_{x \to a^-} f(x) = f(a)$$

- A function $f(x)$ is said to be continuous from the right at $x = a$ if

$$\lim_{x \to a^+} f(x) = f(a)$$

- A function $f(x)$ is said to be continuous on the open interval (a, b) if it is continuous at each & every point of that interval.
- A function is said to be continuous on $[a, b]$

  if (i) Continuous on (a, b)

  (ii) Right continuous at $x = a$

  (iii) Left continuous at $x = b$
- A function $f(x)$ is said to be discontinuous at $x = a$ if it is not continuous there.
- Different reasons for discontinuity :

  (i) $f(a)$ does not exist

  (ii) $\lim_{x \to a} f(x)$ does not exist

  (iii) Both $\lim_{x \to a} f(x)$ and $f(a)$ exist but different from each other.
- **Discontinuity of First kind :**

  If right hand limit and left hand limit exist but are unequal i.e. $\lim_{x \to a^+} f(x) \neq \lim_{x \to a^-} f(x)$
- **Discontinuity of Second kind :**

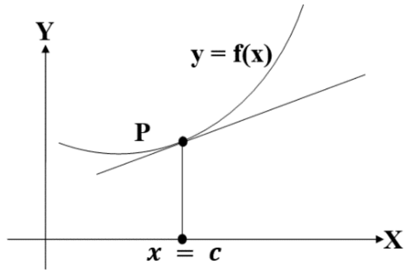
  Neither $\lim_{x \to a^+} f(x)$ exist nor $\lim_{x \to a^-} f(x)$ exist
- **Removal Discontinuity :**

  If $\lim_{x \to a} f(x)$ exists but not equal to $f(a)$.
- If f is continuous at $x = a$ and g is continuous at $f(a)$ then gof is continuous at $x = a$.

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 12

DERIVATIVE



- Let $y = f(x)$ be a given curve, then the slope of the tangent to this curve at $x = c$ is given by

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \left[\frac{df(x)}{dx}\right]_{x=c}$$

- Derivative of $f(x)$ is defined by

$$f'(x) = \frac{df(x)}{dx} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$$

- Derivative of $y = f(x)$ at a point $x = a$, defined by

$$[f'(x)]_{x=a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

- Left hand derivative at $x = a$ defined by

$$f'(a^-) = \lim_{h \to 0} \frac{f(a - h) - f(a)}{-h}$$
$$= \lim_{x \to a^-} \frac{f(x) - f(a)}{x - a}$$

- Right hand derivative at $x = a$ is defined as

$$f'(a^+) = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$
$$= \lim_{x \to a^+} \frac{f(x) - f(a)}{x - a}$$

- A function $f : (a, b) \to \mathbb{R}$ is said to be derivable or differentiable at $x = c \in (a, b)$ if

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} \text{ exists.}$$

- Differentiability $\Rightarrow$ Continuity, but the converse is not true.
- **Algebra of Derivative :**

(i) $\frac{d}{dx}\big(kf(x)\big) = k\frac{d}{dx} f(x)$

(ii) $\frac{d}{dx}\big(f(x) \pm g(x)\big) = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x)$

(iii) $\frac{d}{dx}\big(f(x)g(x)\big) = g(x)f'(x) + f(x)g'(x)$ (Product Rule)

(iv) $\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{g(x)f'(x) - f(x)g'(x)}{\big(g(x)\big)^2}$ (Quotient Rule)

Some Important Formulae :

(i) $\frac{d}{dx}\ x^n = nx^{n-1}, n \in \mathbb{Q}$

(ii) $\frac{d}{dx}\ (c) = 0$ (c is a constant)

(iii) $\frac{d}{dx}\ a^x = a^x \ln a \quad (a > 0)$

(iv) $\frac{d}{dx} e^x = e^x$

(v) $\frac{d}{dx} \log_e x = \frac{1}{x}$

(vi) $\frac{d}{dx} \log_a x = \frac{1}{x \ln a}\ (a > 0)$

(vii) $\frac{d}{dx} \sin x = \cos x$

(viii) $\frac{d}{dx} \cos x = -\sin x$

(ix) $\frac{d}{dx} \tan x = \sec^2 x$

(x) $\frac{d}{dx} \cot x = -\operatorname{cosec}^2 x$

(xi) $\frac{d}{dx} \sec x = \sec x \tan x$

(xii) $\frac{d}{dx} \operatorname{cosec} x = -\operatorname{cosec} x \cot x$

- For a composite function $y = g\big(f(x)\big)$

  We find

  $$\frac{dy}{dx} = \frac{dg\big(f(x)\big)}{df(x)} \cdot \frac{df(x)}{dx}$$

  which is known as **Chain Rule**.

- If $y = f(t),\ x = g(t)$ then

  $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

  This is called as derivative of a function with respect to another function.

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 13

MATHEMATICAL REASONING

- **Statement or Proposition :**

  A declarative sentence which is either true or false is called a statement.

  e.g. (i) $\sqrt{2}$ is a rational number. (False)

  (ii) 3 is greater than 2. (True)

- Following sentences are not statements

  (i) What is your name?

  (ii) May God bless you!

  (iii) x is an alphabet.

- Statements are denoted by letter p, q, r, … etc and their truth value by T (true) or F(false).
- A statement whose truth value does not depend on any other statement is called a **simple statement**.
- The statement which is a combination of two or more simple statements is called a **compound statement**.
- If p is a statement then $\sim p$ (not p) is its negation.
- The word or symbol which is used to form a compound statement is called **connective**.
- Some usual connectives are : and (Λ), or (V), implication (⇒), not (~) negation, both side implication (⇔)
- If p and q are two statements then $p \wedge q$ is called the conjunction statement. It is true only when p as well as q is true, otherwise false.
- If p and q are two statements then $p \vee q$ is called the disjunction statement. It is false only when p as well as q is false, otherwise true.
- The statement $p \rightarrow q$ is false only when p is true and q is false, otherwise true.
- For $p \rightarrow q$, the inverse is $\sim p \rightarrow \sim q$

  the converse is $q \rightarrow p$

  the contrapositive is $\sim q \rightarrow \sim p$

- **Construction of truth table**

e.g.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

- The bi-conditional statement $p \leftrightarrow q$ is true if p and q have same truth values, otherwise false.
- Two compound statements p and q are called equivalent if the truth values of both are same in each case.
- If a compound statement is always true then we call it a **tautology**.
- If a compound statement is always false then we call it a **fallacy**.
- If $p(x)$: x is greater than 5 is a statement then 'x' is called its subject and " is greater then 5" its predicate.
- Sometimes we write

$$\forall\, x\, f(x) \qquad\qquad \{\forall \rightarrow for\ all\}$$

Here $\forall$ is a quantifier.

- Similarly, $\exists\, x\, g(x)$ gives a quantifier $\exists$ (there exists).

Note : Suitable examples and related problems may be incorporated as per necessity.

MATHEMATICAL INDUCTION METHOD

- Suppose we want to show that a statement $p(n)$ is true $\forall\, n \in \mathbb{N}$, then we can prove this by Mathematical Induction Method.
- Mathematical Induction method has following steps :

  (i) Prove that $p(1)$ is true

  (ii) Assume that $p(k)$ is true

  (iii) By the help of $p(k)$ show that $p(k+1)$ is true.
- Example :

  Prove by mathematical induction method :-

$$1+2+3+\cdots+n=\frac{n(n+1)}{2},\ \ n \in \mathbb{N}$$

Proof : Let $p(n):\ 1+2+3+\cdots+n=\frac{n(n+1)}{2}$

(i) For $n=1$, we find

L.H.S. = 1

R.H.S. = $\frac{1(1+1)}{2}$ = 1

(ii) Let $p(k)$ be true, so we have

$$1+2+3+\cdots+k=\frac{k(k+1)}{2}$$

(iii) Now for $n=k+1$, we have

$$\begin{aligned}1+2+3+\cdots+k+(k+1) &= \frac{k(k+1)}{2}+(k+1)\\ &=(k+1)\left(\frac{k}{2}+1\right)\\ &=\frac{(k+1)(k+2)}{2}\\ &=\frac{(k+1)(k+1+1)}{2}\end{aligned}$$

So, $p(k+1)$ is true.

Hence by mathematical induction method $p(n)$ is true $\forall\, n \in \mathbb{N}$.

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 14

PROBABILITY

- **Random Experiment (Trial) :**

  An experiment is called Random Experiment if

  (i) It has more than one possible outcomes.

  (ii) It is not possible to predict the outcome in advance.

- **Outcome :**

  Possible result of a random experiment.

- **Sample Space (S or Ω) :**

  The set of all possible outcomes of a random experiment is called the sample space.

  e.g. If we toss a coin then the possible outcomes are $\{H, T\} = S$ (Sample Space)

- An element of a sample space is called a **Sample Point**.

  Event :

- Any subset of the Sample Space (S) is called an event.
- If E is an event and S is the sample space then $E \subseteq S$

  Compound Event :

- The event which carries more than one event simultaneously is called Compound Event. (In other words – An event containing more than one sample point is called a compound event)

  e.g. Suppose we throw a die, then $S = \{1,\ 2,\ 3,\ 4,\ 5,\ 6\}$

  Let event $E_1 =$ getting an odd number

  $= \{1,\ 3,\ 5\}$

  event $E_2 =$ getting an even number

  $= \{2,\ 4,\ 6\}$

  Then $E_1,\ E_2$ are compound events. $E_1 \cup E_2,\ E_1 \cap E_2$ are also compound events.

- **Impossible Event :**

  If an event (E) contains no sample point, then we call it impossible event.

- Event $E_1 \cup E_2 \to$ The event E_1 or E_2

  $E_1 \cap E_2 \to$ The event E_1 and E_2

  $E_1 - E_2 \to$ The event E_1 but not E_2

- **Mutually Exclusive Events :**

  Two events are mutually exclusive if they have no common element.

  e.g. If we throw a coin once

  and $E_1 \to$ getting head

  $E_2 \to$ getting tail

  Then events $E_1,\ E_2$ are mutually exclusive.

- **Complementary Events :**

  If the sample space is S and E_1 is an event then $E_1' = S - E_1 = E_1^c$ is called the complementary event of E_1

- **Exhaustive Events :**

  The events are called exhaustive if their union gives the sample space.

- **Mutually Exclusive and Exhaustive Events :**

  If the events $A_1, A_2, \dots, A_n$ associated with sample space S

  s.t. $A_1 \cup A_2 \cup \dots \cup A_n = S$ and $A_i \cap A_j = \phi$ for $i, j = 1, 2, \dots, n$

  Then we call these exhaustive and mutually exclusive.

- **Equally Likely Outcomes :**

  Two outcomes are called equally likely if they have equal chance of occurrence.

PROBABILITY

- If S is the sample space and E is an event then probability of E is $P(E) = \frac{|E|}{|S|}$
- Probability of an impossible event is zero.
- **Sure Event :**

  If an event $E = S$ then we call it a sure event.

  For sure event $P(E) = P(S) = 1$

Fundamental Theorems of Probability :

- If A is an event and S is the sample space then $0 \leq P(A) \leq 1$
- If A and B are two events in the sample space S then

  (i) $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

  (ii) $P(A \cup B) = P(A) + P(B)$, if A and B are mutually exclusive.

  (Addition Theorem of Probability)

- For three events A, B, C

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(C \cap A) + P(A \cap B \cap C)$$

- If A and B are two events s.t. $A \subseteq B$ then $P(A) \leq P(B)$
- If A and B are two events then $P(A - B) = P(A) - P(A \cap B)$
- For any event A, $P(A^c) = P(not\ A) = 1 - P(A)$
- **Axiomatic Approach to Probability :**

  If S is the sample space of a random experiment, the probability P is a real function whose domain is the power set of S and range is $[0,\ 1]$ satisfying

  (i) For any event A, $0 \leq P(A) \leq 1$

  (ii) $P(S) = 1$

  (iii) $P(A \cup B) = P(A) + P(B)$, if A and B are mutually exclusive events
- If $A_1, A_2, \dots, A_n$ are pairwise mutually exclusive events then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \cdots + P(A_n)$$

- If the sample space S contains the outcomes $X_1, X_2, \dots, X_n$ then from the axiomatic approach we find

  (i) $0 \leq P(X_i) \leq 1$ for each $X_i \in S$

  (ii) $P(X_1) + P(X_2) + \cdots + P(X_n) = 1$

  (iii) For any event A, $P(A) = \sum P(X_i)$ where $X_i \in A$

Independent Events :

- Two events A and B are called independent if $P(A \cap B) = P(A).P(B)$
- If A and B are independent events then probability of B is not influenced by A.

Finite Probability Space :

- If $S = \{a_1, a_2, \dots, a_n\}$ is a finite sample space then a finite probability space can be obtained by assigning probability p_i to a_i $(i = 1, 2, \dots, n)$ s.t.

  (i) Each $p_i \geq 0$

  (ii) $p_1 + p_2 + \cdots + p_n = 1$
- If an event $A = \{a_1, a_2, \dots, a_r\}$ then

$$P(A) = \sum_{i=1}^{r} p(a_i) = \sum_{i=1}^{r} p_i$$

Equiprobable Space :

- If each sample point in a finite probability space has the same probability, then the probability space is called an equiprobable space.
- A sample space which is not equiprobable is called a non-uniform space.

Note : Suitable examples and related problems may be incorporated as per necessity.

MODULE - 15

STATISTICS

- **Variable :**

  A quantity which has measurable characteristic is called a variable.

  In statistics we call it variate.

- Variables are of 2 types

  (i) Discrete (ii) Continuous

- **Discrete Variable :**

  The variable which takes integral value is called discrete variable.

- **Continuous Variable :**

  The variable which takes real value is called continuous variable.

- **Population :**

  In statistics the word population means a collection of measurements of a given variable.

  A population is said to be finite or infinite according as the given variable is discrete or continuous.

- **Sample :**

  A population or a selected portion of it is called a sample.

  Sample is a subset of population.

- **Frequency :**

  The number of times a value of a variable occurs in a population is called the frequency of that variable.

- A data can be presented in 3 ways

  (i) Simple Distribution :

  2, 2, 5, 6, 6, 8, 9, 9, 9, 10

  (Mark obtained by 10 students where the total mark is 10)

(ii) Frequency Distribution :

Score	Frequency
2	2
5	1
6	2
8	1
9	3
10	1
	10

(iii) Grouped Frequency Distribution :

Group or Class	Frequency
0 – 5	3
5 – 10	7

Here the class size is 5.

- **Class Interval :**

  Each group in a grouped frequency distribution is called a class interval.

- **Class Size :**

  The difference between upper limit and lower limit of a class is called class size.

- **Class Mid Value :**

  The average of the lower and upper limit of a class is called as the mid value.

- **Measure of Central Tendency :**

  Generally a huge data is represented by a single value (statistical average) for better understanding. To get such average we use some methods which are called as measure of central tendency. Commonly used methods are :

  (i) Mean (ii) Median (iii) Mode

MEAN :

- Means are of 4 types :

  (i) Arithmetic (ii) Geometric (iii) Harmonic (iv) Weighted

- In a simple distribution $x_1, x_2, \dots, x_n$

  A.M. $= \frac{x_1+x_2+\cdots+x_n}{n} = \bar{X} = \frac{\sum_{i=1}^{n} x_i}{n}$

- If $x_1, x_2, \dots, x_n$ are with frequencies $f_1, f_2, \dots, f_n$ then A.M.$(\bar{X}) = \frac{\sum_{i=1}^{n} f_i x_i}{\sum_{i=1}^{n} f_i}$

- For a grouped frequency distribution, if $x_1, x_2, \ldots, x_n$ are mid-values of the class with frequencies $f_1, f_2, \ldots, f_n$ then A.M.$(\bar{X}) = \frac{\sum f_i x_i}{\sum f_i}$

GEOMETRIC MEAN :

- If $x_1, x_2, \ldots, x_n$ are the scores then G.M. = $(x_1 . x_2 . \ldots . x_n)^{\frac{1}{n}}$
- If $x_1, x_2, \ldots, x_n$ occur with frequencies $f_1, f_2, \ldots, f_n$ then

  G.M. = $\left(x_1^{f_1} . x_2^{f_2} . \ldots . x_n^{f_n}\right)^{\frac{1}{f_1 + f_2 + \cdots + f_n}}$

HARMONIC MEAN :

- If $x_1, x_2, \ldots, x_n$ are the scores then H.M. = $\frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$
- If $x_1, x_2, \ldots, x_n$ occur with frequencies $f_1, f_2, \ldots, f_n$ then

  H.M. = $\frac{f_1 + f_2 + \cdots + f_n}{\left(\frac{f_1}{x_1} + \frac{f_2}{x_2} + \cdots + \frac{f_n}{x_n}\right)}$

WEIGHTED MEAN :

- If $x_1, x_2, \ldots, x_n$ are associated with weights $w_1, w_2, \ldots, w_n$ then weighted mean is defined as W.M. $= \frac{\sum w_i x_i}{\sum w_i}$

MEDIAN :

- It is defined as the value of the variable which divides the data in such a manner that the number of items below and above it are equal provided they are either in ascending or descending order of magnitude.

  (i) If the number of observations is odd (n) then the median is the value of the $\left(\frac{n+1}{2}\right) th$ observation.

  (ii) If the number of observations is even (n) then the median is the average of $\left(\frac{n}{2}\right) th$ and $\left(\frac{n}{2} + 1\right) th$ observations.

- For a grouped frequency distribution

  Median = $l_m + \frac{C\left(\frac{N}{2} - Cf_{m-1}\right)}{f_m}$

  where l_m → lower limit of the median class

  C → Class size

  N → Total number of observations

  f_m → Frequency of the median class

  Cf_{m-1} → Cumulative frequency of the class just precedes the median class

MODE :

- The mode is the value which occurs most often.
- If each score occurs once or in equal frequencies then the data has no mode.
- If two scores occur with highest number of frequencies then the data has bi-modal value.
- For simple frequency distribution Mode can be obtained by observing the score with highest frequency in the data.
- For a grouped frequency distribution

  Mode = $l_m + \frac{C(f_m - f_{m-1})}{(f_m - f_{m-1}) + (f_m - f_{m+1})}$

  where l_m → lower limit of the modal class

  C → Class size of the modal class

  f_m → Frequency of the modal class

  f_{m-1} → Frequency of the class just precedes the modal class

  f_{m+1} → Frequency of the class just succeeds the modal class
- **Empirical Formula :**

  Mode = 3 Median – 2 Mean (A.M.)

MEASURES OF DISPERSION :

- Generally there are 4 types of measures of dispersion :

  (i) Range (ii) Mean Deviation (iii) Standard Deviation (iv) Quartile Deviation

MEAN DEVIATION :

- For simple data $x_1, x_2, \dots, x_n$

  M.D. $= \frac{\sum_{i=1}^{n}|x_i - \bar{X}|}{n}$ $\quad (\bar{X} = A.M.)$

- For the data $x_1, x_2, \dots, x_n$ with frequencies $f_1, f_2, \dots, f_n$

  M.D. = $\frac{\sum_{i=1}^{n} f_i |x_i - \bar{X}|}{\sum_{i=1}^{n} f_i}$

- For grouped frequency distribution

  M.D. = $\frac{\sum f_i |x_i - \bar{X}|}{\sum f_i}$

  where $x_1, x_2, \dots, x_n$ are mid values of the classes.

STANDARD DEVIATION :

- For simple data $x_1, x_2, \dots, x_n$ if $\bar{X}$ is the A.M. then

$$\sigma = S.D. = \sqrt{\frac{\sum_{i=1}^{n}(x_i - \bar{X})^2}{n}}$$

- If $x_1, x_2, \dots, x_n$ occur with frequencies $f_1, f_2, \dots, f_n$ then

$$S.D. = \sqrt{\frac{\sum_{i=1}^{n} f_i (x_i - \bar{X})^2}{\sum_{i=1}^{n} f_i}}$$

VARIANCE :

- The square of the standard deviation is known as variance and denoted by $V = \sigma^2$
- The coefficient of variance is defined as $C.V. = \frac{\sigma}{\bar{X}} \times 100$

  where $\bar{X}$ is the arithmetic mean.

Note : Suitable examples and related problems may be incorporated as per necessity.

Class - XII

MATHEMATICS

CONTENTS

Module

ABSTRACT

Relation & Function :

Revision of relation chapter that covered in 1st year (Upto equivalence class). How relations are combined i.e. $R_1 \cup R_2, R_1 \cap R_2, R_1 - R_2$ etc. Idea about n-ary relation, Idea on recurrence relation.

Revision of function chapter, Composition of functions, Inverse of a function, Binary operations, Types of Binary operations.

Inverse Trigonometric Functions :

Inverse Trigonometric functions with domain and range. Graphs of inverse trigonometric function. Identities involving inverse trigonometric function. Solution of equations containing inverse trigonometric function. Principal value.

Linear Programming :

Introduction on Linear Programming Problems (L.P.P). Definition of objective function, constraints, optimization, non-negative restriction, variables. Formulation of LPP. Graphical method of solution of LPP in two variables. Feasible solution & infeasible solution, unbounded solution of LPP. Infinite solution of LPP. Introduction on Simplex method.

Matrix :

Concept, notation, order, equality, types of matrices, null and identity matrix, transpose of a matrix. Symmetric & Skew Symmetric matrices, Operation on matrices – Addition, Subtraction, multiplication of a matrix with a scalar, Multiplication of two matrices.

Determinant :

Determinant of a square matrix (upto order 3×3), minor, cofactor, properties of determinant, product of determinants. Factorization of determinant, solution of a system of equations by Cramer's rule.

Specific Topics : Adjoint of a matix, Inverse of a matrix, solution of a system of equations by matrix method. Elementary row operation.

Probability :

Revision of 1^{st} year courses, multiplication theorem of probability, Baye's theorem, Probability distribution of random variable, mean and variance of random variable, Bernoulli trials, Binomial distribution.

Limit, Continuity & Differentiability :

Revision of limit & Continuity, Revision of differentiability (Up to chain rule). Derivative of implicit function, Derivative of functions in parametric form, Derivative of $y = [f(x)]^{g(x)}$ using logarithm. Successive derivatives, Derivatives of nth order, some standard formulae like y_n for $y = \sin(ax+b)$, $y = \cos(ax+b)$, $y = a^{cx+d}$, $y = e^{ax+b}$, $y = \log_e(ax+b)$ etc. Several variable functions & partial derivatives, Euler's theorem, Lagrange Mean Value Theorem (Statement only), Rolle's Theorem (Statement only) & their geometrical interpretation.

Application of Derivative :

Rate of change of a function, velocity & acceleration of a moving body. Increasing and decreasing functions, equation of tangent & normal to a given curve. Use of derivative in approximation, Maxima, Minima (1^{st}& 2^{nd} derivative test), point of inflexion. Concavity & Convexity. Application of Maxima & Minima.

Integrals :

Idea of integration as an inverse process of differentiation, indefinite integrals & its notation. Some standard formulae. Algebra of integrals, integration by substitution, integration by trigonometric substitution, integration by partial fraction, integration by parts.

Some special kinds of integration like :

$$\int \frac{dx}{x^2+a^2}, \int \frac{dx}{\sqrt{x^2-a^2}}, \int \frac{dx}{\sqrt{a^2-x^2}}, \int \frac{dx}{ax^2+bx+c}, \int \frac{px+q}{ax^2+bx+c}\ dx,$$

$$\int \sqrt{a^2-x^2}\, dx, \int \sqrt{x^2-a^2}\, dx, \int \sqrt{ax^2+bx+c}\ dx,$$

$$\int (px+q)\sqrt{ax^2+bx+c}\, dx\ etc.$$

Reduction formula. Some other typical integrations.

Definite Integrals :

Definite integrals, expressing integration as a limit of a sum, properties of definite integrals, Mean value theorem for integrals, fundamental theorems of integral calculus, Walli’s formulae. Finding of area of a plane region using definite integrals. Improper integrals.

Differential Equation :

Formation of differential equation, order & degree, general solution, particular solution, singular solution, solution of a differential equation by using variable separable method. Homogeneous differential equation, Exact differential equation. Equation reducible to homogeneous form. Integrating factors, Linear differential equation, Bernoulli differential equation, Solution of 2nd order differential equation. Initial value problem & Boundary value problem.

Vector :

Vector and scalar, magnitude & direction of a vector, types of vectors, Addition and subtraction of vectors, Multiplication of a vector by a scalar, position vector, Division formula (vector form), Resolution of a vector into components, Direction cosines and Direction ratios of a vector, scalar product, properties of scalar product, vector product & its properties, scalar triple product & its properties, vector triple product, product of four vectors.

3-Dimensional Coordinate Geometry :

Introduction, review of distance and division formulae, direction cosines and direction ratios of a straight line, angle between two lines, Plane (introduction), Equation of a plane (Simple form), Equation of a plane - General form, Vector equation of a plane, Intercept form & normal form of a plane, Distance of a point from a plane, equation of planes passing through the line of intersection of two given planes, angle between two planes, Bisector planes. The straight line, Equation of a straight line – symmetric & un-symmetric form, vector equation of a straight line, angle between two lines, Distance of a point from a line, Shortest distance between two lines, angle between a line and a plane, condition for a line that lies on a plane, skew lines. Introduction on sphere, general equation of a sphere, Idea on Cone & Cylinder in 3 – D.

***

LESSON PLAN

(For +2 II Year)

Unit I	Topic To Be Covered	Number of Classes Required	Total Classes
Relation & Function	Revision of relation chapter that covered in 1st year (Upto equivalence class)	1	5
	How relations are combined i.e. $R_1 \cup R_2$, $R_1 \cap R_2$, $R_1 - R_2$ etc. Idea about n - ary relation.	1	
	Idea on recurrence relation with examples.	1	
	Revision of function chapter, Composition of functions, Inverse of a function with examples.	1	
	Binary operations. Types of Binary operations with examples.	1	
	Doubt Clearing Class	1	2
	Problem	1	
Inverse Trigonometric Functions	Inverse Trigonometric functions with domain and range. Graphs of inverse trigonometric function.	1	3
	Identities involving inverse trigonometric function. Solution of equations containing inverse trigonometric function. Principal value.	2	
	Doubt Clearing Class	1	3
	Problem	2	

Linear Programming	Introduction on Linear Programming Problems (L.P.P). Definition of objective function, variables, constraints, optimization, non-negative restriction.	1	4
	Formulation of LPP. Graphical method of solution of LPP in two variables.	1	
	Feasible solution & infeasible solution, unbounded solution of LPP. Infinite solution of LPP.	1	
	Introduction on Simplex method.	1	
	Doubt Clearing Class	1	3
	Problem	1	
	Problem	1	

Unit II	Topic To Be Covered	Number of Classes Required	Total Classes
Matrix	Concept of matrix, notation, order, equality, types of matrices, null and identity matrix	1	3
	Transpose of a matrix. Symmetric & Skew Symmetric matrices, Operation on matrices – Addition, Subtraction, multiplication of a matrix with a scalar, Multiplication of two matrices.	2	
	Doubt Clearing Class	1	2
	Problem	1	

Determinant	Determinant of a square matrix (upto order 3×3), minor, cofactor, properties of determinant	1	4
	Product of determinants. Factorization of determinant, solution of a system of equations by Cramer's rule.	1	
	Specific Topics : Adjoint of a matix, Inverse of a matrix, solution of a system of equations by matrix method.	1	
	Elementary row operation.	1	
	Doubt Clearing Class	1	5
	Doubt Clearing Class	1	
	Problem	1	
	Problem	1	
	Problem	1	
Probability	Revision of 1st year courses	1	3
	multiplication theorem of probability, Baye's theorem, Probability distribution of random variable	1	
	Mean and variance of random variable, Bernoulli trials, Binomial distribution.	1	
	Doubt Clearing Class	1	3
	Problem	1	
	Problem	1	

Unit III	Topic To Be Covered	Number of Classes Required	Total Classes
Limit, Continuity & Differentiability	Revision of limit, Continuity	2	4
	Revision of differentiability (upto chain rule). Differentiability of a function at a particular point, Derivative of implicit function, Derivative of functions in parametric form	2	
	Problem	1	2
	Problem	1	
	Derivative of $y = [f(x)]^{g(x)}$ using logarithm. Successive derivatives, Derivatives of nth order, some standard formulae like y_n for $y = \sin(ax+b)$, $y = \cos(ax+b)$, $y = a^{cx+d}$, $y = e^{ax+b}$, $y = \log_e(ax+b)$ etc.	2	2
	Problem	1	2
	Problem	1	
	Several variable functions & Partial derivatives, Euler's theorem.	2	3
	Lagrange mean value theorem (Statement only) & Rolle's Theorem(Statement only) with their geometrical interpretation.	1	
	Doubt Clearing Class	1	3
	Problem	1	
	Problem	1	

Application of Derivative	Rate of change of a function, velocity & acceleration of a moving body.	1	2
	Increasing and decreasing functions, equation of tangent & normal to a given curve.	1	
	Doubt Clearing Class	1	2
	Problem	1	
	Use of derivative in approximation, Maxima, Minima (1^{st}& 2^{nd} derivative test), point of inflexion.	1	2
	Concavity & Convexity. Application of Maxima &Minima.	1	
	Doubt Clearing Class	1	2
	Problem	1	

Unit IV	Topic To Be Covered	Number of Classes Required	Total Classes
Integrals	Idea of integration as an inverse process of differentiation, indefinite integrals & its notation. Some standard formulae.	1	2
	Algebra of integrals, integration by substitution, integration by trigonometric substitution	1	
	Doubt Clearing Class	1	2
	Problem	1	
	Integration by partial fraction, integration by parts.	2	2

	Doubt Clearing Class	1	2
	Problem	1	
	Some special kinds of integration like : $\int \frac{dx}{x^2+a^2}, \int \frac{dx}{\sqrt{x^2-a^2}}, \int \frac{dx}{\sqrt{a^2-x^2}}, \int \frac{dx}{ax^2+bx+c}, \int \frac{px+q}{ax^2+bx+c}\ dx$	1	1
	Problem	2	2
	Some special kinds of integration like : $\int \sqrt{a^2-x^2}\,dx, \int \sqrt{x^2-a^2}\,dx, \int \sqrt{ax^2+bx+c}\ dx,$ $\int (px+q)\sqrt{ax^2+bx+c}\,dx$	1	1
	Problem	2	2
	Reduction formulae. Some other typical integrations.	1	1
	Problem	1	1
Definite Integrals	Definite integrals, expressing integration as a limit of a sum, properties of definite integrals	1	1
	Problem	1	1
	Mean value theorem for integrals, fundamental theorems of integral calculus, Walli's formulae.	1	2
	Finding of area of a plane region using definite integrals. Improper integrals	1	
	Doubt Clearing Class	1	2
	Problem	1	

Differential Equation	Formation of differential equation, order & degree, general solution, particular solution, singular solution,	1	3
	Solution of a differential equation by using variable separable method. Homogeneous differential equation, Exact differential equation.	2	
	Doubt Clearing Class	1	2
	Problem	1	
	Equation reducible to homogeneous form. Integrating factors, linear differential equation, Bernoulli differential equation	2	2
	Problem	1	1
	Solution of 2^{nd} order differential equation. Initial value problem & Boundary value problem.	1	1
	Problem	1	1

Unit V	Topic To Be Covered	Number of Classes Required	Total Classes
Vector	Vector and scalar, magnitude & direction of a vector, types of vectors, Addition and subtraction of vectors, Multiplication of a vector by a scalar, position vector	1	1
	Problem	1	1
	Division formula (vector form), Resolution of a vector into components, Direction cosines and Direction ratios of a vector, scalar product, properties of scalar product	1	1

	Problem	1	1
	Vector product & its properties, scalar triple product & its properties, vector triple product, product of four vectors.	1	1
	Doubt Clearing Class	1	2
	Problem	1	
3-Dimensional Co-ordinate Geometry	Introduction, review of distance and division formulae, direction cosines and direction ratios of a straight line	1	2
	Angle between two lines, Projection formula	1	
	Doubt Clearing Class	1	2
	Problem	1	
	Plane (introduction), Equation of a plane (Simple form), Equation of a plane - General form, Vector equation of a plane, Intercept form & normal form of a plane	1	1
	Problem	1	1
	Distance of a point from a plane, Equation of a plane through three given points, Equation of planes passing through the line of intersection of two given planes	2	2
	Problem	1	1
	Angle between two planes, Bisector planes	1	1
	Problem	1	1

	The straight line, Equation of a straight line - symmetric & un-symmetric form, Vector equation of a straight line, Angle between two lines	1	1
	Problem	1	1
	Distance of a point from a line, Shortest distance between two lines	1	1
	Problem	1	1
	Angle between a line and a plane, condition for a line that lies on a plane, Skew line.	1	1
	Problem	1	1
	Introduction on sphere, general equation of a sphere, Idea on Cone & Cylinder.	1	1
	Doubt Clearing Class	1	2
	Problem	1	

LESSON PLAN

(For +2 IIYear)

Overall Revision	**Number of Classes Required**
Unit I	3
Unit II	3
Unit III	3
Unit IV	3
Unit V	3

***

MODULE – 1

Relation

- **Cartesian Product :** $A \times B = \{(x, y): x \in A, y \in B\}$
- **Relation :** Any subset of $A \times B$ is a relation from A to B.
- ϕ is the smallest and $A \times B$ is the largest relation from A to B.
- **Domain and Range of Relation :**

  If $R : A \to B$ is a relation then the domain of R is denoted by $dom\ R$ and is defined by

  $$dom\ R = \{x \in A : (x, y) \in R\}$$

  If $R : A \to B$ is a relation then the range of R is denoted by $rng\ R$ and is defined by

  $$rng\ R = \{y \in B : (x, y) \in R\}$$

- **Inverse of a Relation :** Let $R : A \to B$ be a relation. The inverse of R is denoted by R^{-1} and it is a relation from B to A defined by

  $$R^{-1} = \{(y, x) : (x, y) \in R\}$$

- **Identity Relation on a Set :** Identity Relation on a set A is denoted by I_A and defined by

  $$I_A = \{(x, x) : x \in A\}$$

- **Universal Relation on a Set :** Universal Relation on a set A is denoted by U_A and is defined by $U_A = A \times A$.
- **Equivalence Relation :** A relation $f : A \to A$ is said to be an equivalence relation if it is reflexive, symmetric and transitive.

Function

- A relation $f : A \to B$ is a **function** from A to B if

  (i) $dom\ f = A$

  (ii) the relation is either One-One or Many-One.

- **Equality of two Functions :** Two functions f and g are said to be equal iff

  1. They have same domain
  2. They have same range
  3. $f(x) = g(x)\ \forall x \in dom\ f$

Some Important Functions :

- **Characteristic Function :** Let X be any set and A is a subset of X. The characteristic function of A, denoted by X_A , is defined by

$$X_A\,(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}$$

- **Remainder Function :** Let m be a positive integer. Then remainder function is defined by $r_m(n)$ = remainder obtained by dividing $n \in Z$ by m.
- **Binary Operations :** Let A be any non-empty set. A binary operation on A is a function from $A \times A$ to A. Binary operation associates each element of $A \times A$ to a unique element of A.

  Example : Addition, subtraction, multiplication and division are binary operations on R.
- **Commutative :** A binary operation $*$ on a set A is said to be commutative if

  $a * b = b * a \quad \forall\, a, b \in A$

  Example : Addition is commutative but subtraction is not commutative on R
- **Associative :** A binary operation $*$ on a set A is said to be associative if

  $a * (b * c) = (a * b) * c \quad \forall\, a, b, c \in A$

  Example : Addition is associative, but subtraction is not associative on R.
- **Existence of Identity Element :** Let $*$ be a binary operation on a set A. Then A is said to have an identity element if there exists a unique element e in A such that

  $a * e = a = e * a \qquad \forall\ a \in A.$

  Note :

  0 is the additive identity element in R and 1 is the multiplicative identity element in R.
- **Existence of Inverse Element :** Let $*$ be a binary operation on a set A. Let a be any element of A. Then a is said to have an inverse element, if there exists a unique element b in A such that $a * b = e = b * a$.
- **Combining Relation :**

  Since a relation from A to B is a subset of $A \times B$, we can combine two relations as two sets.

  e.g. Let $A = \{1, 2, 3\}, B = \{1, 2, 3, 4\}$

  The relation $R_1 = \{(1, 1), (2, 2), (3, 3)\}$

  $R_2 = \{(1, 1), (1, 2), (1,3), (1, 4)\}$

  can be combined as below :

  $R_1 \cup R_2 = \{(1, 1), (1, 2), (1,3), (1, 4), (2, 2), (3, 3)\}$

$R_1 \cap R_2 = \{(1,1)\}$

$R_1 - R_2 = \{(2,2),(3,3)\}$

$R_2 - R_1 = \{(1,2),(1,3),(1,4)\}$

The above are also relations from A to B.

- **n-ary Relation :**

  Let $A_1, A_2, \dots, A_n$ be sets.

  An n–ary relation on these sets is a subset of $A_1 \times A_2 \times \dots \times A_n$.

  Here sets $A_1, A_2, \dots, A_n$ are called domains of the relation and **n** is called its degree.

e.g. Let R be a relation on $N \times N \times N$ consisting of triples (a, b, c)

where a, b, c are +ve integers with $a < b < c$. Then $(1,2,3) \in R$ but $(2,4,3) \notin R$.

The degree of the relation is 3. Its domains are all equal to the set of natural numbers.

- **Recurrence Relation :**

A recurrence relation for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely $a_0, a_1, a_2, \dots, a_{n-1}$ for all integers n with $n \geq n_0$ where n_0 is a non-negative integer.

e.g. Let $\{a_n\}$ be a sequence that satisfies the recurrence relation $a_n = a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, \dots$ and suppose $a_0 = 3, a_1 = 5$ then what are a_2 and a_5 ?

Solution : Here $a_0 = 3, a_1 = 5$

Then $a_2 = a_1 - a_0 = 5 - 3 = 2$

$a_3 = a_2 - a_1 = 2 - 5 = -3$

$a_4 = a_3 - a_2 = -3 - 2 = -5$

$a_5 = a_4 - a_3 = -5 - (-3) = -2$

MODULE - 2

INVERSE TRIGONOMETRIC FUNCTIONS

- **Inverse Trigonometric Functions :**

  $\sin^{-1} : [-1,1] \to \left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ is defined by $y = \sin^{-1} x \Leftrightarrow x = \sin y$

  $\cos^{-1} : [-1,1] \to [0,\pi]$ is defined by $y = \cos^{-1} x \Leftrightarrow x = \cos y$

  $\tan^{-1} : R \to \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ is defined by $y = \tan^{-1} x \Leftrightarrow x = \tan y$

  $\cot^{-1} : R \to (0,\pi)$ is defined by $y = \cot^{-1} x \Leftrightarrow x = \cot y$

  $\sec^{-1} : (-\infty,-1] \cup [1,\infty) \to \left[0,\frac{\pi}{2}\right) \cup \left(\frac{\pi}{2},\pi\right]$ is defined by $y = \sec^{-1} x \Leftrightarrow x = \sec y$

  $\operatorname{cosec}^{-1} : (-\infty,-1] \cup [1,\infty) \to \left[-\frac{\pi}{2},0\right) \cup \left(0,\frac{\pi}{2}\right]$ is defined by

  $y = \operatorname{cosec}^{-1} x \Leftrightarrow x = \operatorname{cosec} y$

- **Important Identities :**

  $$\sin^{-1} x = \operatorname{cosec}^{-1}\frac{1}{x}$$

  $$\cos^{-1} x = \sec^{-1}\frac{1}{x}$$

  $$\tan^{-1} x = \cot^{-1}\frac{1}{x}$$

  $$\cot^{-1} x = \tan^{-1}\frac{1}{x}$$

  $$\sec^{-1} x = \cos^{-1}\frac{1}{x}$$

  $$\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$$

- $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$

  $$\tan^{-1} x + \cot^{-1} x = \frac{\pi}{2}$$

  $$\sec^{-1} x + \operatorname{cosec}^{-1} x = \frac{\pi}{2}$$

- $\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

  $$\tan^{-1} x - \tan^{-1} y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$$

  $$\tan^{-1} x + \tan^{-1} y + \tan^{-1} z = \tan^{-1}\left(\frac{x+y+z-xyz}{1-xy-yz-zx}\right)$$

- $\sin^{-1} x \pm \sin^{-1} y = \sin^{-1}\left[x\sqrt{1-y^2} \pm y\sqrt{1-x^2}\right]$

  $\cos^{-1} x \pm \cos^{-1} y = \cos^{-1}\left[xy \mp \sqrt{(1-x^2)(1-y^2)}\right]$
- $2\sin^{-1} x = \sin^{-1}\left(2x\sqrt{1-x^2}\right)$

  $2\cos^{-1} x = \cos^{-1}(2x^2-1)$

  $2\tan^{-1} x = \sin^{-1}\left(\dfrac{2x}{1+x^2}\right) = \cos^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) = \tan^{-1}\left(\dfrac{2x}{1-x^2}\right)$
- $3\sin^{-1} x = \sin^{-1}(3x-4x^3)$

  $3\cos^{-1} x = \cos^{-1}(4x^3-3x)$

  $3\tan^{-1} x = \tan^{-1}\left(\dfrac{3x-x^3}{1-3x^2}\right)$
- $\sin^{-1}(\sin y) = y$ if $y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

  $\sin(\sin^{-1} x) = x$ if $x \in [-1, 1]$

  $\cos^{-1}(\cos y) = y$ if $y \in [0, \pi]$

  $\cos(\cos^{-1} x) = x$ if $x \in [-1, 1]$

  $\tan^{-1}(\tan y) = y$ if $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

  $\tan(\tan^{-1} x) = x$ if $x \in (-\infty, \infty)$
- **Principal Value :**

  We know $\sin\frac{\pi}{6} = \frac{1}{2}$, $\sin\frac{5\pi}{6} = \frac{1}{2}$, ...

  So $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$, $\frac{5\pi}{6}$, ...

  As $\frac{\pi}{6} \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, we say $\frac{\pi}{6}$ is the principal value of $\sin^{-1}\left(\frac{1}{2}\right)$.

  Note : The value of the inverse trigonometric function which lies in the range is called the principal value.
- **Solution of Equation with Inverse Trigonometric Functions :**

  e.g. Solve $\tan^{-1} x + \tan^{-1}\frac{2x}{1+x^2} = \frac{\pi}{2}$

  Solution : Putting $x = \tan\theta$, we find

$$\tan^{-1} x + \tan^{-1}\frac{2x}{1+x^2} = \tan^{-1}(\tan\theta) + \tan^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$= \theta + 2\theta = 3\theta = \frac{\pi}{2}$$

  So, $\theta = \frac{\pi}{6}$ and hence $x = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}$

MODULE - 3

LINEAR PROGRAMMING PROBLEM (L.P.P)

- To find $x_1, x_2, \dots, x_n$ which optimizes $z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$ (1)

  Subject to

$$\left.\begin{array}{c} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n (\leq = \geq) b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n (\leq = \geq) b_2 \\ \vdots \quad\quad \vdots \quad\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n (\leq = \geq) b_m \end{array}\right\} \quad (2)$$

  and $x_1, x_2, \dots, x_n \geq 0$ (3)

- This can be written as

  Optimize $z = cx$

  subject to $Ax(\leq = \geq)b$ and $x \geq 0$

  where $c = [c_1\ c_2 \dots c_n]$, $x = [x_1\ x_2 \dots x_n]^T$, $A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}$

- **Objective Function :** The function (1) is known as the Objective Function.
- **Constraints :** The equations and in-equations (2) are called the Constraints.
- **Decision Variables :** The variables $x_1, x_2, \dots, x_n$ are called the decision variables.
- **Cost Coefficients :** The constants $c_1, c_2, \dots, c_n$ are called the Cost Coefficients.
- **Non-negative Restrictions :** The restriction in (3) are called the non-negative restrictions.
- **Feasible Solution :** The set of values of variables which satisfies the set of constraints and non-negative restrictions of the L.P.P. is called the feasible solution of that L.P.P.
- **Optimum Feasible Solution :** The feasible solution which optimizes the objective function of the L.P.P. is called the optimum feasible solution of that L.P.P.
- **Graphical Solution of L.P.P. :**

  L.P.P. involving two variables can be solved easily by graphical method.
- **Working procedure to solve a L.P.P. graphically**

  Step – I : Converting all in-equations of the constraints as equations and then drawing the straight lines correspond to those equations we can get the necessary feasible region.

Step – II : Now we have to determine the vertices of the feasible region.

Step – III : Now we have to evaluate the value of the objective function at each vertex.

Step – IV : The vertex which gives the optimum (maximum or minimum) value of the objective function will be the optimal solution to the problem.

- Solve graphically :-

Max $z = 2x + 5y$

Subject to $x + y \geq 4$

$6x + y \leq 12$

and $x, y \geq 0$

Solution :

(i) Let us convert all the constraints into equations, so we get

$$x + y = 4, \qquad 6x + y = 12$$

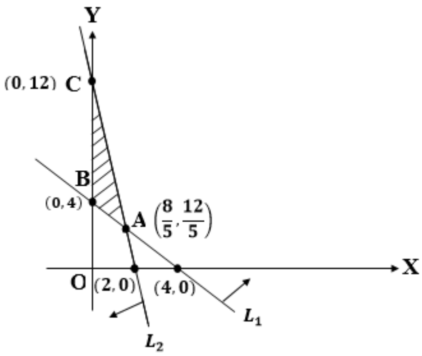
(ii) Let us draw straight lines correspond to the above lines :-

$L_1 : x + y = 4$

X	0	4
Y	4	0

$L_2 : 6x + y = 12$

x	0	2
y	12	0



(iii) Here the shaded region ABC is the feasible region.

(iv) At $A\left(\frac{8}{5}, \frac{12}{5}\right)$ we get $z = 2\left(\frac{8}{5}\right) + 5\left(\frac{12}{5}\right) = \frac{16}{5} + 12 = \frac{76}{5}$

At $B(0, 4)$ we get $z = 2(0) + 5(4) = 20$

At $C(0, 12)$ we get $z = 2(0) + 5(12) = 60$

So, max $z = 60$, where $x = 0, y = 12$

- **Unbounded Solution :**

e.g. Solve graphically :-

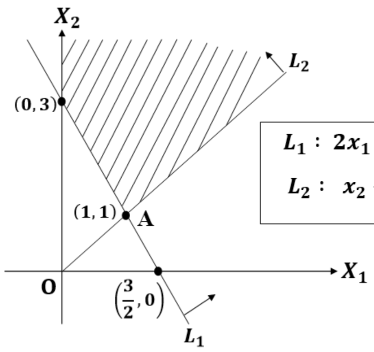
Max $z = 6x_1 + x_2$

Subject to $2x_1 + x_2 \geq 3$

$x_2 - x_1 \geq 0$

and $x_1, x_2 \geq 0$

Graph



Here feasible solution exists but max(z) is unbounded.

- **Infeasible Solution :**

e.g. Solve graphically :-

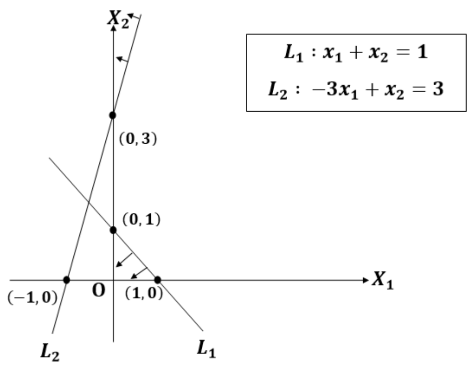
Max $z = x_1 + x_2$

Subject to $x_1 + x_2 \leq 1$

$-3x_1 + x_2 \geq 3$

and $x_1, x_2 \geq 0$

Graph



There is no feasible region which satisfies both the constraints. So this LPP has infeasible solution.

- **Simplex Method :**

  This is an iterative procedure for solving a linear programming problem in a finite number of steps.

- Generally, we use simplex method to solve the LPP which contains more than 2 variables.

MODULE - 4

MATRIX

- A matrix is an arrangement of elements in certain number of rows and columns. A matrix is denoted by English capital letter.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdot & \cdot & \cdot & a_{1n} \\ a_{21} & a_{22} & \cdot & \cdot & \cdot & a_{2n} \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & & \cdot \\ a_{m1} & a_{m2} & \cdot & \cdot & \cdot & a_{mn} \end{bmatrix}$$

  where the $(i,j)^{th}$ elements of matrix A are denoted as a_{ij} . Horizontal line is called as **Row** and vertical line as **Column**.

- **Order of matrix :** If a matrix A has m rows and n columns then order of the matrix A is $m \times n$ that is $A = [a_{ij}]_{m\times n}$
- **Row Matrix :** A matrix with a single row is called a row matrix, i.e. order of the row matrix is $1 \times n$.

  e.g. $A = [1\ 9\ 7]_{1\times 3}$
- **Column Matrix :** A matrix with a single column is called a column matrix, i.e. order of the column matrix is $n \times 1$.

  e.g. $A = \begin{bmatrix} 1 \\ 7 \\ 9 \end{bmatrix}_{3\times 1}$
- **Square matrix :** A matrix in which number of rows is equal to number of columns is called a square matrix, i.e. order of the square matrix is $n \times n$.
- **Diagonal Matrix :** A square matrix in which the non-diagonal elements are all zero is called a diagonal matrix.

  e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$
- **Scalar Matrix :** A diagonal matrix in which the diagonal elements are all equal is called a scalar matrix.

  e.g. $A = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$

- **Unit Matrix (Identity Matrix) :** A diagonal matrix in which the diagonal elements are all 1 is called a unit matrix.

  e.g. $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

- **Zero Matrix :** A matrix in which the elements are all equal to zero is called a Zero matrix or Null matrix and usually denoted by O.

- **Algebra of Matrices :**

  (i) **Addition :** If $A = \left(a_{ij}\right)_{m\times n}$ and $B = \left(b_{ij}\right)_{m\times n}$ then $A + B = \left(a_{ij} + b_{ij}\right)_{m\times n}$

  (ii) **Subtraction :** If $A = \left(a_{ij}\right)_{m\times n}$ and $B = \left(b_{ij}\right)_{m\times n}$ then $A - B = \left(a_{ij} - b_{ij}\right)_{m\times n}$

  (iii) **Multiplication :**

  Multiplication of Matrix by a Scalar :

  If $A = \left(a_{ij}\right)_{m\times n}$ and k is a scalar then $kA = \left(ka_{ij}\right)_{m\times n}$

  Multiplication of a Matrix by a Matrix :

  If $A = \left(a_{ij}\right)_{m\times n}$ and $B = \left(b_{ij}\right)_{n\times p}$ then only multiplication of matrices is possible and it is defined by $AB = \left(c_{ij}\right)_{m\times p}$ where $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$

- **Equality of Two Matrices :** Two matrices A and B are said to be equal iff

  (i) Order of A and B are same.

  (ii) Each element of A is equal to the corresponding element of B.

- **Transpose of a Matrix :** If in a matrix A, we interchange all rows into columns and vice-versa, then the new matrix so obtained is called transpose of the matrix A and is denoted by A^T or A'.

  e.g. $if\ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\ then\ A' = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$

- **Symmetric Matrix :** A square matrix A is said to be a symmetric matrix if $A = A'$.

  Example : $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} = A'$

- **Skew Symmetric Matrix :** A square matrix A is said to be a skew symmetric matrix if $A = -A'$.

  Example : $A = \begin{bmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{bmatrix} = -A'$

- Every square matrix can be uniquely expressed as a sum of symmetric and a skew symmetric matrix.

$$\text{i.e.,}\quad A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

- **Elementary Row Operations :**
    1. Interchanging two rows.
    2. Multiplying a row by a non-zero scalar.
    3. Adding to a row a scalar times another row.

  The above operations are useful to find the inverse of a given matrix and to solve a given set of linear equations.
- When a matrix A is subjected to a finite number of elementary row operations, the resulting matrix B is said to be row equivalent to A and is denoted by $B \sim A$.

MODULE - 5

DETERMINANT

- It is defined as the value that can be computed from the elements of a square matrix A.
- We can also define determinant as follows :

  Let us solve $a_1x + b_1y = c_1$

  $a_2x + b_2y = c_2$

  Solution : Here

$$x = \frac{c_1b_2 - b_1c_2}{a_1b_2 - a_2b_1}\ , y = \frac{-(c_1a_2 - a_1c_2)}{a_1b_2 - a_2b_1}$$

  and both exist if $a_1b_2 - a_2b_1 \neq 0$. This value can be written as

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$$

  and the L.H.S. is called a determinant of 2^{nd} order.
- Let A be a square matrix of order 2 such that $A = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$

  Then $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ is called the determinant of A.

  Symbolically, $det(A) = |A|$

  Here $|A| = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1$ is the value of the determinant.

- Similarly, let A be a square matrix of order 3 such that $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

  Then $|A| = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$

  $$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

- **Properties :**

  1. The values of determinant remains unchanged by changing row to columns and columns to rows.
  2. The inter change of two adjacent rows or columns of a determinant changes the sign of determinant without changing its numerical value.
  3. If two rows or columns of a determinant are identical then the value of the determinant is zero.
  4. If every element of any row or column is multiplied by a factor then the determinant is multiplied by that factor.
  5. A determinant remains unchanged by adding k times the elements of any row (or columns) to corresponding elements of another row (or column), where k is any number.
  6. If every element of any row (or column) of a determinant be expressed as sum of two numbers then the determinant can be expressed as sum of two determinants.

- **Minors and Cofactors :**

  Let A be a square matrix of order 3 given by

  $$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

  Then the **Minor** of $a_{ij} = M_{ij} =$ value of the determinant obtained by omitting i^{th} row and j^{th} column and the **Cofactor** of a_{ij} is defined by $C_{ij} = (-1)^{i+j} M_{ij}$

- **Singular Matrix :** A square matrix whose determinant value equals to zero is called a singular matrix.

- **Non-Singular Matrix :** A square matrix which is not singular is called a non-singular matrix.

- **Cramer's Rule :**

  To solve the following systems of linear equations

  $a_1x + b_1y + c_1z = k_1$

  $a_2x + b_2y + c_2z = k_2$

  $a_3x + b_3y + c_3z = k_3$

  We consider $\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \Delta_x = \begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$,

  $\Delta_y = \begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}, \Delta_z = \begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$

  Then we find

  $$x = \frac{\Delta_x}{\Delta}, \quad y = \frac{\Delta_y}{\Delta}, \quad z = \frac{\Delta_z}{\Delta}$$

- **Notes :**
  1. If $\Delta \neq 0$ then the system has only one solution.
  2. If $\Delta = 0$ and atleast one of $\Delta_x, \Delta_y, \Delta_z$ is not zero then the system has no solution.
  3. If $\Delta = \Delta_x = \Delta_y = \Delta_z = 0$ then the system has infinite number of solutions.

- **Consistent System of Equations :**

  A system of equations is said to be consistent if it has solution.

- **Inconsistent System of Equations :**

  A system of equations is said to be inconsistent if it has no solution.

Specific uses of Matrix and Determinant :

- **Adjoint of a Matrix :** The transpose of the co-factor matrix of a given matrix A is called its adjoint and denoted by **adj A**.

- **Inverse of a Matrix :**

  If A is a non-singular matrix then its inverse matrix (A^{-1}) is defined as

  $$A^{-1} = \frac{adj\ A}{|A|}$$

- **Solution of a System of Linear Equations :**

  Let the system of linear equations be

  $a_1x + b_1y + c_1z = d_1$

  $a_2x + b_2y + c_2z = d_2$

  $a_3x + b_3y + c_3z = d_3$

$$\Rightarrow \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

$\Rightarrow AX = b$

$\Rightarrow X = A^{-1}b$

$\Rightarrow X = \dfrac{adj\ A}{|A|}.b$

- **Note :** We can find the inverse of a non-singular matrix by using elementary row operations.

MODULE - 6

PROBABILITY

- **Outcome :** The results of an experiment are called outcomes.
- **Random Experiment :** An experiment is called random experiment if its outcome are uncertain.
- **Sample Space :** The set of all possible outcomes of a random experiment is called the sample space and is denoted by S or Ω.
- **Sample Point :** An element of a sample space is called a sample point.
- **Event :** Any subset of a sample space is called an event.
- **Probability :** The probability of an event A is denoted by P(A) and defined by $P(A) = \frac{|A|}{|S|}$, where S is the sample space.
- **Conditional probability :** Let B be an event. Then the conditional probability of B relative to event A is denoted by $P(B/A)$ and is defined by

$$P(B/A) = \frac{P(B \cap A)}{P(A)}$$

  Note :

  This is the probability of event B when event A has already been occurred.
- If A and B are independent events, then $P(A \cap B) = P(A)P(B)$
- If A and B are not independent, then $P(A \cap B) = P(A)P(B/A)$ (Another form of conditional probability)
- **Extension of Conditional Probability :** $P(A \cap B \cap C) = P(A)P(B/A)P(C/A \cap B)$

- **Law of Total Probability :**

  Let S be the sample space and let $E_1, E_2, \dots, E_n$ be **n** mutually exclusive and exhaustive events associated with a random experiment. If A be any event which occurs with E_1 or E_2 or … or E_n then $P(A) = P(E_1)\, P(A/E_1) + P(E_2)\, P(A/E_2) + \cdots + P(E_n)\, P(A/E_n)$

- **Baye's Theorem :**

  Let S be the sample space and let $E_1, E_2, \dots, E_n$ be n mutually exclusive and exhaustive events associated with a random experiment. If A be any event which occurs with E_1 or E_2 or … or E_n then

  $$P(E_i/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)}, \text{where i} = 1, 2, \dots, \text{n.}$$

- **Random Variable (X) :** A random variable is a numerically valued function defined on the sample space S. It is a rule that assigns a numerical value to each possible outcome of an experiment.

- **Discrete Random Variable :**

  A discrete random variable is a variable which can only take a countable number of values.

  e.g. The number of students in each +2 college of Odisha.

- **Continuous Random Variable :**

  The random variable which takes the value from the set of Real numbers is called a continuous random variable.

  e.g. The height of each student in B.J.B. College.

- **Probability Mass Function :**

  Let X be a discrete random variable, which takes the possible values $x_1, x_2, \dots, x_n$.

  With each x_i we associate a real number

  $P_i = P(X = x_i)$, i = 1, 2, …, n satisfying the following conditions :

  (i) $P_i \geq 0$ for each i

  (ii) $\sum_{i=1}^{n} P_i = 1$

  Then the function $P_i = P(X = x_i)$ is called the probability mass function.

  The set of all ordered pairs $(x, P(x))$ is called the probability distribution of X.

- **Mean and Variance :**

  Let X be a random variable having the following probability distribution :

  $x : \; x_1 x_2 \ldots x_n$

  $P(x) : P_1 P_2 \ldots \; P_n$

  Mean of random variable $X \; = \; \bar{x} = \; \sum_{i=1}^{n} x_i \, P(x_i)$

  Variance of random variable $X = \sigma^2 = \; \sum_{i=1}^{n} (x_i - \bar{x})^2 \, P(x_i)$

  $$= \sum_{i=1}^{n} x_i^2 \, P(x_i) - (\bar{x})^2$$

  The positive square root of the variance of X is called the standard deviation of X and is denoted by σ.

- **Trial :** Each time an experiment if performed is a trial.
- **Bernoulli Trial :** Trials of a random experiment are called Bernoulli trial if the following conditions are satisfied :

  (1) The number of trial is finite

  (2) Trials are independent

  (3) The outcomes are dichotomous (success or failure)

  (4) The probability of success (or failure) in each trial is constant.

- **Binomial Distribution :**

  If the number of trials is n, the probability of a success in each trial = p and the probability of a failure in each trial = q, then

  Probability(r success in n trials) = $^{n}C_r p^r q^{n-r}$

  Mean $= \sum_{r=0}^{n} r p(r) = np$

  Variance = $\sum r^2 \, p(r) - \left(\sum r p(r)\right)^2 = npq$

MODULE - 7

LIMIT

- For a function $f(x)$ we have $\lim_{x \to a} f(x) = l$ means $f(x) \to l$ whenever $x \to a$.

  Analytically, $\lim_{x \to a} f(x) = l$ iff for every $\epsilon > 0$, there exists $\delta > 0$ (depending on ϵ) such that $|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$.

- **Left Hand Limit and Right Hand Limit :**

  (i) L is said to be left hand limit of $f(x)$ as $x \to a$

  i.e., $\lim_{x \to a^-} f(x) = L$ iff for every $\epsilon > 0$ there exists $\delta > 0$

  such that $|f(x) - l| < \epsilon$ whenever $a - \delta < x < \delta$.

  (ii) L is said to be right hand limit of $f(x)$ as $x \to a$

  i.e., $\lim_{x \to a^+} f(x) = L$ iff for every $\epsilon > 0$ there exists $\delta > 0$

  such that $|f(x) - l| < \epsilon$ whenever $\delta < x < a + \delta$.

- $\lim_{x \to \infty} x^n = \begin{cases} \infty & if\ n > 0 \\ 1 & if\ n = 0 \\ 0 & if\ n < 0 \end{cases}$

- $\lim_{n \to \infty} x^n = \begin{cases} 0 & if\ |x| < 1 \\ 1 & if\ x = 1 \\ \infty & if\ x > 1 \\ does\ not\ exist & if\ x \leq -1 \end{cases}$

- **L'Hospital Rule :**

  Let $\lim_{x \to a} f(x) = 0 = \lim_{x \to a} g(x)$ and f and g are differentiable at $x = a$ then

  $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

  Note : This can be repeatedly used up to a finite steps until we find a particular value of the limit.

CONTINUITY

- A function f is said to be continuous at $x = a$ if

  (i) f is defined at a

  (ii) $\lim_{x \to a} f(x)$ exists i.e., $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x)$

  (iii) $\lim_{x \to a} f(x) = f(a)$

- If a function f is not continuous at $x = a$ then we call it discontinuous at that point.

MODULE - 8

DIFFERENTIATION

- Let $y = f(x)$ be a function of x defined in an interval (a, b).

  Then the differential coefficient (derivative) of y or $f(x)$ with respect to x (denoted as $\frac{dy}{dx}$ or $f'(x)$ or $Df(x)$) is defined by

$$\frac{dy}{dx} = \lim_{\mathrm{h}\to 0} \frac{f(\mathrm{x}+\mathrm{h}) - f(x)}{\mathrm{h}}$$

Differentiable at a Point and Differentiability of a Function :

- A function $y = f(x)$ is said to be differentiable at $x = a$ if $\lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists.

  Note :

  Here $f(x)$ is defined on an interval $[\alpha, \beta] \subset R$ and $\alpha < a < \beta$.

- $\lim_{x\to a^-} \frac{f(x)-f(a)}{x-a}$ is called the left hand derivative and $\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a}$ is called the right hand derivative of y or $f(x)$ at $x = a$.
- **Note :** $y = f(x)$ is differentiable at $x = a$ if L.H.D. = R.H.D. at $x = a$.
- **Derivative of Inverse Trigonometric Functions :**

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\mathrm{cosec}^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

- Let $y = f(x)$ be a function and $f'(x)$ exists then

  $y_1 = \frac{dy}{dx} = f'(x)$ is called the first order derivative of y with respect to x.

  $y_2 = \frac{d^2y}{dx^2} = f''(x)$ is called the second order derivative of y with respect to x.

$y_n = \frac{d^n y}{dx^n} = f^n(x)$ is called the n^{th} order derivative of y with respect to x.

- **Some Standard Results :**

1. $$\frac{d^n}{dx^n}(ax+b)^m = \begin{cases} \frac{m!\,a^n(ax+b)^{m-n}}{(m-n)!} & ,if\ m>n \\ m!\,a^m & ,if\ m=n \\ 0 & ,if\ m<n \end{cases}$$
2. $$\frac{d^n}{dx^n}\sin(ax+b) = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$$
3. $$\frac{d^n}{dx^n}\cos(ax+b) = a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$$
4. $$\frac{d^n}{dx^n}\log(ax+b) = (-1)^n\,(n-1)!\,a^n(ax+b)^{-n}$$
5. $$\frac{d^n}{dx^n}a^x = a^x\,(\log_e a)^n$$

- **Leibnitz Theorem :**

If u and v are differentiable functions having n^{th} derivative then

$$\frac{d^n}{dx^n}(uv) = {}^nC_0u_nv_0 + {}^nC_1u_{n-1}v_1 + {}^nC_2u_{n-2}v_2 + \cdots + {}^nC_nu_0v_n$$

Here $u_i = \frac{d^i}{dx^i}(u)$ and $v_i = \frac{d^i}{dx^i}(v)$, co-efficient ${}^nC_r = \frac{n!}{r!(n-r)!}$

Several Variable Function and Partial Derivative :

- If a function has more than one independent variables we call it several variable function.

Mathematically,

$$z = f(x,y)$$
$$u = f(x,y,z)$$
$$y = f(x_1,x_2,\dots,x_n)$$

e.g. $u = 2x^2 + 3y + z$

$v = (x^2+y^2)^{\frac{3}{4}}$

- For the function

$y = f(x_1,x_2,\dots,x_n)$ we find $\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2}, \dots, \frac{\partial y}{\partial x_n}$ as partial derivatives.

Note : $\frac{\partial y}{\partial x_1}$ means the simple derivative of y w.r.t. x_1 where other variables $x_2, x_3, \dots, x_n$ are treated as constant.

- **Successive Partial Derivative :**

  For $z = f(x, y)$

  $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$$

  $$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$$

  $$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) \ or \ \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$$

  are known as successive partial derivatives.

- A several variable function $y = f(x_1, x_2, \dots, x_n)$ is called **homogeneous of degree k** if we can express $y = t^k f(u_1, u_2, \dots, u_n)$, where $x_1 = tu_1, x_2 = tu_2, \dots, x_n = tu_n$

- **Euler's Theorem :** If $f(x, y)$ is a homogeneous function of degree **n** then

  $$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

Special Note :

- If

  $$F(x) = \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix}$$

  Then

$$F'(x) = \begin{vmatrix} f_1'(x) & f_2'(x) & f_3'(x) \\ g_1(x) & g_2(x) & g_3(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ {g_1}'(x) & {g_2}'(x) & {g_3}'(x) \\ h_1(x) & h_2(x) & h_3(x) \end{vmatrix} + \begin{vmatrix} f_1(x) & f_2(x) & f_3(x) \\ g_1(x) & g_2(x) & g_3(x) \\ {h_1}'(x) & {h_2}'(x) & {h_3}'(x) \end{vmatrix}$$

MODULE - 9

APPLICATION OF DERIVATIVE

- **Uses of Derivatives :**
  1. Equation of Tangent line to a curve $y = f(x)$ at the point $(x_1,\ y_1)$ is given by

$$y - y_1 = \left(\frac{dy}{dx}\right)_{(x_1, y_1)} (x - x_1)$$

Equation of Normal line to a curve $y = f(x)$ at the point $(x_1,\ y_1)$ is given by

$$y - y_1 = \frac{-1}{\left(\frac{dy}{dx}\right)_{(x_1, y_1)}} (x - x_1)$$

  2. Let $y = f(x)$ be defined in (a, b). Then
     (i) $f'(x) > 0 \ \forall\ x \in (a, b) \Rightarrow f$ is strictly increasing on (a, b).
     (ii) $f'(x) \geq 0 \ \forall\ x \in (a, b) \Rightarrow f$ is monotonically increasing on (a, b).
     (iii) $f'(x) < 0 \ \forall\ x \in (a, b) \Rightarrow f$ is strictly decreasing on (a, b).
     (iv) $f'(x) \leq 0 \ \forall\ x \in (a, b) \Rightarrow f$ is monotonically decreasing on (a, b).
     (v) $f'(x) = 0 \ \forall\ x \in (a, b) \Rightarrow f$ is a constant function on (a, b).

- **Critical Point :**

  Critical point is a point at which the derivative vanishes or does not exist.

- **First Derivative Test to find local Maximum/Minimum :**

  For a function $f(x)$, we can follow the steps given below to evaluate the local maximum or local minimum

  1. Find a point $x = c$ such that $f'(c) = 0$ or $f'(c)$ does not exist (critical points).
  2. If $f'(x)$ changes its sign from $+ve$ to $-ve$ as x passes through the point $x = c$ then local maximum value of $f(x)$ occurs at $x = c$.
  3. If $f'(x)$ changes its sign from $-ve$ to $+ve$ as x passes through the point $x = c$ then local minimum value of $f(x)$ occurs at $x = c$.
  4. If $f'(x)$ does not change its sign as x passes through the point $x = c$ then neither local maximum nor local minimum value of $f(x)$ occurs at $x = c$.

- **Second Derivative Test to find local Maximum / Minimum :**

  For a function $f(x)$, we can follow the steps given below to evaluate the local maximum or local minimum

  1. Find a point $x = c$ such that $f'(c) = 0$ or $f'(c)$ does not exist (critical points).
  2. If $f''(c) < 0$ then local maximum value of $f(x)$ occurs at $x = c$.
  3. If $f''(c) > 0$ then local minimum value of $f(x)$ occurs at $x = c$.
  4. If $f''(c) = 0$ then the test fails and nothing definite can be said about the point $x = c$.

  Note :

  If point (4) is valid then we can proceed up to nth successive derivatives such that $\boldsymbol{f^{(n)}(c) \neq 0}$ $\left(f'(c) = 0 = f''(c) = \cdots = f^{(n-1)}(c)\right)$ and then decide the local maximum or local minimum as per the value of nth derivative is –ve or +ve. However, if $(n-1)$ is even, then f has neither a max. nor a min. at $x = c$.

- **Point of Inflexion :** A point on a curve is said to be a point of inflexion if the curve is concave on one side and convex on the other side of that point with respect to X-axis.

- **Procedure to find Point of inflexion :**

  1. Find a point $x = c$ such that $f''(c) = 0$ or $f''(c)$ does not exist (critical points).
  2. If $f''(x)$ changes its sign as x passes through the point $x = c$ then point of inflexion occurs at $x = c$.
  3. If $f''(x)$ does not change its sign as x passes through the point $x = c$ then point of inflexion does not occur at $x = c$.

- **Rolle's Theorem :**

  If a function f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$ then there exists at least one point c in (a, b) such that $f'(c) = 0$.

- **Lagrange's Mean value Theorem :**

  If a function is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$ then there exists atleast a point c in (a, b) such that

$$\frac{f(b) - f(a)}{(b - a)} = f'(c)$$

MODULE - 10

INTEGRAL CALCULUS

- When $\frac{d}{dx}F(x) = f(x)$ we can get antiderivative as below

$$\int f(x)\, dx = F(x) + C$$

where (i) $\int$ (elongated S) is the sign of integration

(ii) $f(x)$ is the integrand

(iii) C is the constant of integration.

(iv) The process of getting antiderivative or primitive is called **Integration**.

(v) Integration gives the **Integral** of a given function.

Some Standard Integration Formulae :

- $\int x^n\, dx = \frac{x^{n+1}}{n+1} + C,\ where\ n \neq -1$
- $\int \frac{1}{x} dx = ln|x| + C$
- $\int sinx\, dx = -cosx + C$
- $\int cosx\, dx = sinx + C$
- $\int tanx\, dx = ln|secx| + C \quad or \quad - ln|cosx| + C$
- $\int cotx\, dx = ln|sinx| + C \quad or \quad - ln|cosecx| + C$
- $\int secx\, dx = ln|secx + tanx| + C$
- $\int cosecx\, dx = ln|cosecx - cotx| + C$
- $\int sec^2x\, dx = tanx + C$
- $\int cosec^2x\, dx = -cotx + C$
- $\int secx\, tanx\, dx = secx + C$
- $\int cosecx\, cotx\, dx = -cosecx + C$
- $\int e^x\, dx = e^x + C$
- $\int a^x\, dx = \frac{a^x}{lna} + C,\ a > 0$

- $\int \frac{1}{a^2-x^2} dx = \frac{1}{2a} ln \left|\frac{a+x}{a-x}\right| + C$
- $\int \frac{1}{a^2+x^2} dx = \frac{1}{a} tan^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} ln \left|\frac{x-a}{x+a}\right| + C$
- $\int \frac{1}{\sqrt{a^2-x^2}} dx = sin^{-1}\left(\frac{x}{a}\right) + C$
- $\int \frac{1}{\sqrt{x^2+a^2}} dx = ln\left|x + \sqrt{x^2 + a^2}\right| + C$
- $\int \frac{1}{\sqrt{x^2-a^2}} dx = ln\left|x + \sqrt{x^2 - a^2}\right| + C$
- $\int \frac{1}{x\sqrt{x^2-a^2}} dx = \frac{1}{a} sec^{-1}\left(\frac{x}{a}\right) + C$
- $\int \sqrt{a^2 - x^2}\, dx = \frac{1}{2}\left\{x\sqrt{a^2 - x^2} + a^2 sin^{-1}\left(\frac{x}{a}\right)\right\} + C$
- $\int \sqrt{x^2 + a^2}\, dx = \frac{1}{2}\left\{x\sqrt{x^2 + a^2} + a^2 ln\left|x + \sqrt{x^2 + a^2}\right|\right\} + C$
- $\int \sqrt{x^2 - a^2}\, dx = \frac{1}{2}\left\{x\sqrt{x^2 - a^2} - a^2 ln\left|x + \sqrt{x^2 - a^2}\right|\right\} + C$

Algebra of Integration :

- $\int [f(x) \pm g(x)]\, dx = \int f(x)\, dx \pm \int g(x)\, dx$
- $\int k. f(x)\, dx = k \int f(x)\, dx$

Integration by Parts :

- $\int f(x) g(x)\, dx = f(x) \int g(x)\, dx - \int \left[\int g(x)\, dx \left[\frac{d}{dx} f(x)\right]\right] dx$

In simple form we can write integration by parts as below

$$\int u\, dv = uv - \int v\, du$$

where (i) u is the first function and v is the second function.

(ii) du and dv imply derivatives of u and v respectively.

(iii) We can use a formula named as ILATE to determine the first and second function.

$(I \rightarrow Inverse, L \rightarrow Logarithm, A \rightarrow Algebraic, T \rightarrow Trigonometric, E \rightarrow Exponential$

Special Case :

$$\int [f(x) + f'(x)]\, e^x dx = f(x) e^x + C$$

Integration by Partial Fraction

- When we have a fraction $\frac{f(x)}{g(x)}$ such that $f(x)$ and $g(x)$ both are polynomials of x, we call it a proper fraction if degree of $f(x)$ < degree of $g(x)$.

  If we have a proper fraction $\frac{f(x)}{g(x)}$, we can split this into a sum of simpler fractions & the method doing this is called partial fraction.

  (1) $\dfrac{f(x)}{(ax+b)(cx+d)} = \dfrac{A}{ax+b} + \dfrac{B}{cx+d}$

  (2) $\dfrac{f(x)}{(ax+b)^n(cx+d)} = \dfrac{A_1}{ax+b} + \dfrac{A_2}{(ax+b)^2} + \cdots + \dfrac{A_n}{(ax+b)^n} + \dfrac{B}{cx+d}$

  (3) $\dfrac{f(x)}{(ax+b)(cx^2+dx+e)} = \dfrac{A}{ax+b} + \dfrac{Bx+F}{cx^2+dx+e}$

  (4) $\dfrac{f(x)}{(cx^2+dx+e)(\alpha x^2+\beta x+\gamma)} = \dfrac{Ax+B}{cx^2+dx+e} + \dfrac{Cx+D}{\alpha x^2+\beta x+\gamma}$

  (5) $\dfrac{f(x)}{(cx^2+dx+e)(\alpha x^2+\beta x+\gamma)^n} = \dfrac{Ax+B}{cx^2+dx+e} + \dfrac{C_1x+D_1}{\alpha x^2+\beta x+\gamma}$

  $$+\frac{C_2x+D_2}{(\alpha x^2+\beta x+\gamma)^2} + \cdots + \frac{C_nx+D_n}{(\alpha x^2+\beta x+\gamma)^n}$$

MODULE – 11

SOME SPECIFIC INTEGRATION

- **Special Type :**

  (i) $\displaystyle\int \frac{dx}{ax^2+bx+c} = \int \frac{dx}{a\left(x+\frac{b}{2a}\right)^2 + \frac{4ac-b^2}{4a^2}}$ (1st step)

  then put $t = x + \dfrac{b}{2a}$

  (ii) For $\displaystyle\int \frac{dx}{\sqrt{ax^2+bx+c}}$,

  we have to take the coefficient of x^2 common and outside the square root.

  After this the given integral reduces to

  $\displaystyle\int \frac{dt}{\sqrt{t^2+a^2}}$ or $\displaystyle\int \frac{dt}{\sqrt{t^2-a^2}}$ or $\displaystyle\int \frac{dt}{\sqrt{a^2-t^2}}$

  by using proper substitution.

$(iii)\ \int \frac{px+q}{ax^2+bx+c}dx = \int \frac{\frac{p}{2a}(2ax+b)+\frac{2aq-bp}{2a}}{ax^2+bx+c}dx$

$(iv)\ \int \sqrt{ax^2+bx+c}\ dx$ can be reduced to $\int \sqrt{t^2+a^2}\,dt$ or $\int \sqrt{t^2-a^2}\,dt$ or $\int \sqrt{a^2-t^2}\,dt$ by taking coefficient of x^2 common & outside the square root & then using proper substitution.

$(v)\ \int e^{ax}\sin(bx+c)\ dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\sin\left(bx+c-\tan^{-1}\frac{b}{a}\right)+\ k$

$(vi)\ \int e^{ax}\cos(bx+c)\ dx = \frac{e^{ax}}{\sqrt{a^2+b^2}}\cos\left(bx+c-\tan^{-1}\frac{b}{a}\right)+\ k$

$(vii)\ \int \frac{f'(x)}{f(x)}\ dx = \log[f(x)]+c$

- Integrals like

  $\int \frac{dx}{a+b\cos x+c\sin x}$ can be evaluated by expressing $sin\,x$ and $cos\,x$ in terms of $tan^2\frac{x}{2}$ and then putting $tan\frac{x}{2}=t$.

- Integrals like

  $\int \frac{p(x)}{f(x)\sqrt{g(x)}}\ dx$ where $f(x)$and $g(x)$ are linear functions of $x,$ can be evaluated by putting $g(x)=\ t^2.$

- For integrals of the form

  $\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$ we put $ax+b=\frac{1}{t}$

- For integrals of the form

  $\int \frac{dx}{(ax^2+b)\sqrt{cx^2+d}}$ we put $x=\frac{1}{t}$ then $c+dt^2=u^2$

MODULE - 12

DEFINITE INTEGRALS

If $\int f(x)\,dx = \phi(x) + C$ then $\int_a^b f(x)\,dx = \phi(b) - \phi(a)$

Algebra of Definite Integral :

- $\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$
- $\int_a^b k.f(x)\,dx = k\int_a^b f(x)\,dx$

Properties of Definite Integral :

- $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$
- $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
- $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad provided\ that\ a < c < b$
- $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$
- $\int_{-a}^a f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & if\ f(-x) = f(x)\ (i.e. the\ function\ is\ even) \\ 0, & if\ f(-x) = -f(x) (i.e. the\ function\ is\ odd) \end{cases}$
- $\int_0^{2a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx, & if\ f(2a-x) = f(x) \\ 0, & if\ f(2a-x) = -f(x) \end{cases}$
- $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

Reduction Formulae :

$$\int sin^n x\,dx = \frac{-cosx\ sin^{n-1}x}{n} + \frac{n-1}{n}\int sin^{n-2}x\,dx$$

$$\int cos^n x\,dx = \frac{sinx\ cos^{n-1}x}{n} + \frac{n-1}{n}\int cos^{n-2}x\,dx$$

WALLI'S Formulae :

$$\int_0^{\frac{\pi}{2}} sin^n x\,dx = \begin{cases} \frac{(n-1)}{n}.\frac{(n-3)}{(n-2)}.\frac{(n-5)}{(n-4)}\cdots\frac{2}{3}, & if\ n\ is\ odd \\ \frac{(n-1)}{n}.\frac{(n-3)}{(n-2)}.\frac{(n-5)}{(n-4)}\cdots\frac{1}{2}.\frac{\pi}{2}, & if\ n\ is\ even \end{cases}$$

$$\int_0^{\frac{\pi}{2}} cos^n x\, dx = \begin{cases} \frac{(n-1)}{n} . \frac{(n-3)}{(n-2)} . \frac{(n-5)}{(n-4)} \cdots \frac{2}{3}, & if\ n\ is\ odd \\ \frac{(n-1)}{n} . \frac{(n-3)}{(n-2)} . \frac{(n-5)}{(n-4)} \cdots \frac{1}{2} . \frac{\pi}{2}, & if\ n\ is\ even \end{cases}$$

Applications (Area of a closed curve):

- If $y = f(x)$ is a continuous function in $[a, b]$, then $\int_a^b f(x)\, dx$ or $\int_a^b y\, dx$ represents the area bounded by the curve, X- axis and ordinates at $x = a, x = b$.
- If $x = g(y)$ is a continuous function in $[a, b]$, then $\int_a^b g(y)\, dy$ or $\int_a^b x\, dy$ represents the area bounded by the curve, Y- axis and abscissa at $y = a, y = b$.

- **Improper Integral :** The integral

$$\int_a^b f(x)\, dx \text{ is called an improper integral, if}$$

(i) $a = -\infty$ or $b = \infty$ or both, i.e. one or both integration limits are infinite.

(ii) $f(x)$ is unbounded at one or more points of $a \leq x \leq b$. Such points are called singularities of $f(x)$.

e.g.

$$\int_0^{\infty} \cos x \;\; dx \quad \rightarrow \text{ improper integral of } 1^{\text{st}} \text{ kind}$$

$$\int_0^{3} \frac{dx}{x-1} \quad \rightarrow \text{ improper integral of } 2^{\text{nd}} \text{ kind}$$

$$\int_0^{\infty} \frac{dx}{(1-x)^2} \quad \rightarrow \text{improper integral of } 3^{\text{rd}} \text{ kind (combination of } 1^{\text{st}} \text{ and } 2^{\text{nd}} \text{ kind)}$$

MODULE - 13

DIFFERENTIAL EQUATIONS

- An equation containing derivatives or differentials is known as a differential equation.

  e.g. (i) $\frac{dy}{dx} = \sin x$

  (ii) $(x + y + 1)\frac{dy}{dx} = x + y + 3$

- **Order :**

  Order of a differential equation is the order of the highest order derivative term in it.

- **Degree :**

  Degree of a differential equation is the power of the highest order derivative term in it, provided the equation can be expressed as a polynomial equation in derivatives.

- **Linear Differential Equations :**

  A differential equation is said to be linear if the dependent variable and its derivative occurring in this equation are of first degree only and are not multiplied together. A differential equation which is not linear is called a non-linear differential equation.

- **Solution of a differential equation :**

  A relation between the variables is called a solution of the differential equation if it reduces the equation to an identity when substituted to it.

- **General Solution :**

  A general solution of a differential equation is a solution which contains as many arbitrary constants as the order of the equation.

- **Particular Solution :**

  A particular solution of a differential equation is a solution obtained from its general solution by assigning particular values to the arbitrary constants.

- Differential equations of the following form can be solved by separation of variables

  (i) $\frac{dy}{dx} = f(x)$

  (ii) $\frac{dy}{dx} = f(x)g(y)$

  (iii) $\frac{dy}{dx} = g(y)$

  (iv) $\frac{d^2y}{dx^2} = f(x)$

- The differential equation $M(x, y)\, dx + N(x, y) dy = 0$ is **exact** if and only if

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

- **Integrating Factor (I.F.) :** When the left side of the differential equation

$$M(x, y)\, dx + N(x, y) dy = 0$$

is not exact or total differential after its multiplication the L.H.S. of the equation becomes exact or express as $d\mu$. This factor $\mu(x, y)$ is called as an integrating factor.

- **Initial Value Problem (IVP) :**

The problem of determining a solution $y(x)$ of the first order differential equation $\frac{dy}{dx} = f(x, y)$ which satisfies the condition $y(x_0) = y_0$, where y_0 is the prescribed constant is called an initial value problem.

- Similarly for finding a solution $y(x)$ of $y'' = f(x, y, y')$ subject to $y(x_0) = y_0, y'(x_0) = y_1$ is called IVP for 2nd order equation.

- **Boundary Value problem :**

For the equation $y'' = f(x, y, y')$ defined on the interval (x_0, x_1), the BVP is to determine a solution $y(x)$ which satisfies

(i) $y(x_0) = \alpha$ and $y(x_1) = \beta$

OR

(ii) $y(x_0) = \alpha$ and $y'(x_1) = b$

OR

(iii) $y'(x_0) = a$ and $y(x_1) = \beta$

where α, β, a, b are prescribed constants.

- To solve an IVP or BVP we first obtain a general solution and then use given conditions to evaluate the arbitrary constants involved in the general solution.

- **General Form of first order Linear Differential Equation :**

(i) $\frac{dy}{dx} + P(x)y = Q(x)$

Integrating Factor = $\int e^{P(x)}\, dx$

Solution is $y \times I.F. = \int Q(x) \times (I.F.)\, dx$

(ii) $\frac{dx}{dy} + P(y)x = Q(y)$

Integrating Factor = $\int e^{P(y)}\, dy$

Solution is $x \times I.F. = \int Q(y) \times (I.F.)\, dy$

- **Bernoulli's Equation :**

$$\frac{dy}{dx} + P(x)y = Q(x)y^n, \qquad n \neq 1$$

This can be reduced to linear equation by dividing y^n in both sides.

- **Homogeneous Differential Equation :**

A differential equation is of the form $\frac{dy}{dx} = \frac{f(x,y)}{g(x,y)}$ where $f(x, y)$ and $g(x, y)$ are homogeneous functions of x, y and of same degree, is said to be a homogeneous differential equation.

Such type of equations can be solved by putting $y = vx$

- **Special Form :**

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

where (i) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$ $\quad or \quad (ii)$ $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$.

Such type of equation can be reduced to a homogeneous form and then solved.

- **Solution of Second Order and First Degree Differential Equation :**

Solve : $\frac{d^2y}{dx^2} = 6x + 5$

Solution : Let $u = \frac{dy}{dx}$

So, $\frac{d^2y}{dx^2} = \frac{du}{dx} = 6x + 5$

$\Rightarrow du = (6x + 5)dx$

Now, integrating both sides, we get

$$\int du = \int (6x + 5)\, dx$$

$$\Rightarrow u = \frac{dy}{dx} = 3x^2 + 5x + c_1$$

$\Rightarrow dy = (3x^2 + 5x + c_1)dx$

Again, integrating both sides, we get

$$\int dy = \int (3x^2 + 5x + c_1)\, dx$$

$$\Rightarrow y = x^3 + \frac{5}{2}x^2 + c_1x + c_2$$

which is the required general solution.

MODULE - 14

THREE DIMENSIONAL CO-ORDINATE GEOMETRY

- **Distance Formula :**

  The distance between the points $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ is given by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Division Formula :

- If $R(x, y, z)$ divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ internally in the ratio $K : 1$, then $x = \frac{Kx_2+x_1}{K+1},\ y = \frac{Ky_2+y_1}{K+1},\ z = \frac{Kz_2+z_1}{K+1}$
- If $R(x, y, z)$ divides the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ externally in the ratio $K : 1$, then $x = \frac{Kx_2-x_1}{K-1},\ y = \frac{Ky_2-y_1}{K-1},\ z = \frac{Kz_2-z_1}{K-1}$
- The mid-point of the line segment PQ is $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$

Direction Cosines and Direction Ratios :

- If a line L makes angle α, β, γ with +ve direction of X, Y and Z axes respectively, then α, β, γ are called the direction angles of L.
- $cos\alpha, cos\beta, cos\gamma$ are the direction cosines (d.c.s) of L.
- Generally, we denote $l = cos\alpha, m = cos\beta, n = cos\gamma$
- The numbers which are proportional to the d.c.s of a line are called its direction ratios (d.r.s). i.e. if l, m, n are d.c.s of a line L and $\frac{l}{a} = \frac{m}{b} = \frac{n}{c}$,

  then $< a, b, c >$ are called the d.r.s. of L.

Note :

(i) $l^2 + m^2 + n^2 = 1$, where l, m, n are the d.c.s of any line.

(ii) d.c.s of a line are unique.

(iii) d.r.s of a line are infinitely many.

(iv) If $< a, b, c >$ are d.r.s of a line then its d.c.s are

$$< \frac{a}{\pm\sqrt{a^2 + b^2 + c^2}}, \frac{b}{\pm\sqrt{a^2 + b^2 + c^2}}, \quad \frac{c}{\pm\sqrt{a^2 + b^2 + c^2}} >$$

(v) If $P(x, y, z)$ is any point on the line OP whose dcs are $< l, m, n >$ and $|OP| = r$ then $x = lr, y = mr, z = nr$

Here O represents the origin $(0, 0, 0)$

(vi) The d.r.s of the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are given by

$$< x_2 - x_1, y_2 - y_1, z_2 - z_1 >$$

Angle Between Two Lines :

- Angle θ between two lines whose d.c.s are $< l_1, m_1, n_1 >$ and $< l_2, m_2, n_2 >$ is given by $cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$
- If $< a_1, b_1, c_1 >$ and $< a_2, b_2, c_2 >$ are their d.r.s then

$$cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

Note :

(i) Two lines are parallel iff $l_1 = l_2,\ m_1 = m_2,\ n_1 = n_2\ \ or\ \ \frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(ii) Two lines are perpendicular iff $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$$or\ \ a_1 a_2 + b_1 b_2 + c_1 c_2 = 0$$

THE PLANE

- **Definition :**

A plane is a surface where the line joining any two points on it lies totally on the surface.

Equation of a Plane :

- **One Point Form :**

The equation of a plane which passes through (x_0, y_0, z_0) and whose normal has d.c.s (l, m, n) is given by $l(x - x_0) + m(y - y_0) + n(z - z_0) = 0$.

Similarly, if $< a, b, c >$ represents the d.r.s of the normal, then the equation of the plane is $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

- **Three Point Form :**

The plane passes through $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) has the equation

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix} = 0$$

- **Normal Form :**

The equation of plane in normal form is $lx + my + nz = P$

where (l, m, n) are the d.c.s of the perpendicular drawn to the plane from the origin and P is the perpendicular distance of the plane from the origin.

- **Intercept Form :**

The equation of the plane in intercept form is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

where $a =$ X- intercept, $b =$ Y- intercept, $c =$ Z- intercept of the plane.

- **General Form :**

  The general form of equation of a plane is $Ax + By + Cz + D = 0$

  where $< A, B, C >$ are d.r.s of a normal to the plane.

- **Properties :**

  Angle θ between two planes $A_1x + B_1y + C_1z + D_1 = 0$ and

  $A_2x + B_2y + C_2z + D_2 = 0$ is given by

$$cos\theta = \frac{A_1A_2 + B_1B_2 + C_1C_2}{\pm\sqrt{A_1^2 + B_1^2 + C_1^2}\sqrt{A_2^2 + B_2^2 + C_2^2}}$$

Note :

- These two planes are parallel if $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$

  perpendicular if $A_1A_2 + B_1B_2 + C_1C_2 = 0$ and

  identical if $\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}$

- The normal form of $Ax + By + Cz + D = 0$ is represented by

$$\frac{Ax}{\pm\sqrt{A^2 + B^2 + C^2}} + \frac{By}{\pm\sqrt{A^2 + B^2 + C^2}} + \frac{Cz}{\pm\sqrt{A^2 + B^2 + C^2}} = \frac{-D}{\pm\sqrt{A^2 + B^2 + C^2}}$$

  where one of signs + or – will be taken to make R.H.S. (+ve)

- Distance of a plane $Ax + By + Cz + D = 0$ from any point (x_0, y_0, z_0) is

$$d = \left|\frac{Ax_0 + By_0 + Cz_0 + D}{\pm\sqrt{A^2 + B^2 + C^2}}\right|$$

- Equation of planes bisecting the angle between two planes

  $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ are given by

$$\frac{A_1x + B_1y + C_1z + D_1}{\sqrt{A^2 + B^2 + C^2}} = \pm\frac{A_2x + B_2y + C_2z + D_2}{\sqrt{A^2 + B^2 + C^2}}$$

MODULE - 15

THE STRAIGHT LINE

- A straight line in space is the set of all points of intersection of two non-parallel planes.
- **Equation of the Straight Line :**

  If a straight line is obtained by the intersection of two planes $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ then we can represent that line by giving both the equations simultaneously.

  This type of representation is called as Un-symmetric form.
- **Symmetric Form (One Point Form) :**

  The equation of the straight line which passes through (x_0, y_0, z_0) and whose d,r,s are $< a, b, c >$ is represented by $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$
- **Tow Point Form :**

  The equation of the straight line through (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}$$

Properties :

- The line $\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$ will lie on the plane $Ax + By + Cz + D = 0$ if

  (i) $Aa + Bb + Cc = 0$

  (ii) $Ax_0 + By_0 + Cz_0 + D = 0$
- Two lines $\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1}$ and $\frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$ are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

- **Skew Lines :**

  Two lines in space are said to be skew if they are neither parallel nor intersecting.

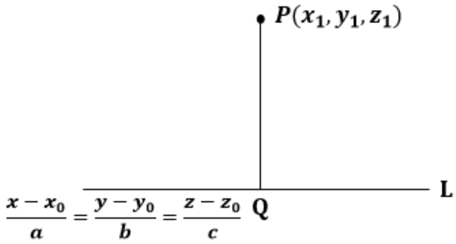
- **Shortest Distance of a Line From a Point :**

  Procedure :

  Let $\dfrac{x-x_0}{a}=\dfrac{y-y_0}{b}=\dfrac{z-z_0}{c}=\lambda$

  So $x=a\lambda+x_0, y=b\lambda+y_0, z=c\lambda+z_0$

  Let Q has co-ordinates $(a\lambda+x_0, b\lambda+y_0, c\lambda+z_0)$.



  Now we can find d.r.s of PQ (Here P is (x_1, y_1, z_1)). Since $PQ \perp L$ we can use the condition of perpendicularity for the lines PQ & L with known d.r.s.

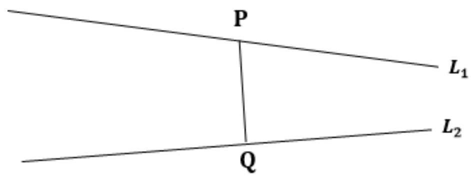
  From this we can get λ then the exact coordinates of Q & then the distance $|PQ|$

- Angle θ between the Line $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$ and a plane $Ax+By+Cz+D=0$ is given by $\theta=sin^{-1}\left[\pm\frac{aA+bB+cC}{\sqrt{a^2+b^2+c^2}\sqrt{A^2+B^2+C^2}}\right]$

- **Shortest Distance Between Two Lines :**

  Let the lines are $L_1: \frac{x-x_1}{a_1}=\frac{y-y_1}{b_1}=\frac{z-z_1}{c_1}$ & $L_2: \frac{x-x_2}{a_2}=\frac{y-y_2}{b_2}=\frac{z-z_2}{c_2}$

  Suppose $L_1=\lambda_1$ & $L_2=\lambda_2$ then we can get $(x_1+a_1\lambda_1,\ y_1+b_1\lambda_1,\ z_1+c_1\lambda_1)$ as the co-ordinates of P and similarly we can get $(x_2+a_2\lambda_2,\ y_2+b_2\lambda_2,\ z_2+c_2\lambda_2)$ as the co-ordinates of Q.



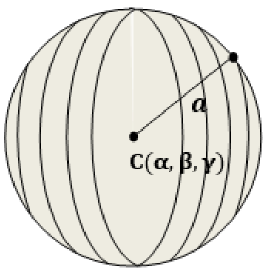
  From these we will get d.r.s of PQ. Now $L_1 \perp PQ$ gives an equation in λ_1 & λ_2 and $L_2 \perp PQ$ gives another equation in λ_1 & λ_2.

  Solving those equation we will get exact values of λ_1 & λ_2. Now we can find exact coordinates of P, Q and then the distance PQ. This is the required shortest distance between two lines.

<u>SPHERE</u>

- A sphere is a surface in $\mathbb{R}^3$, all of whose points are at a constant distance from a fixed point.

  Note : The fixed point is the centre & constant distance is the radius of the sphere.



- If (α, β, γ) is the centre and 'a' is the radius of a sphere then its equation is given by

  $(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=a^2$

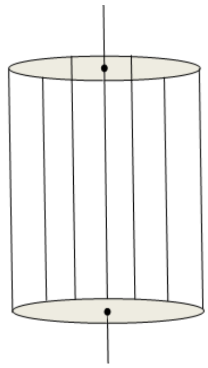
- General equation of a sphere is given by $x^2+\ y^2+z^2+2ux+2vy+2wz+d=0$

CONE

- The surface generated by a variable ray emerging from a fixed point C and intersecting a given curve Γ is called a cone.
- Here C is the vertex, Γ is the guiding curve and the variable ray is the generator of the cone.

CYLINDER

- A cylinder is a surface generated by a variable line which is parallel to a fixed line L and fulfilling one of the following conditions :
  - (i) The variable line intersects a given curve Γ
  - (ii) The variable line touches a given surface S.
- The variable line is called the generator, the fixed line L is called the axis of the cylinder. The curve Γ is called the guiding curve.



MODULE - 16

VECTORS

- **Scalar** is a physical quantity which has only magnitude.
  e.g. Mass, Time, etc.
- **Vector** is a physical quantity which has both magnitude and direction.
  e.g. Force, Velocity, Acceleration, etc.

  Types of Vectors
- **Unit Vector :**
  A vector whose magnitude is 1 is called unit vector.
- **Zero Vector :**
  A vector whose magnitude is zero and direction is arbitrary called a zero vector or a null vector. It is denoted by $\vec{0}$.
- **Proper Vector :**
  A vector whose magnitude is not zero is called a proper vector.

- **Parallel Vectors :**

  Two vectors $\vec{a}$ and $\vec{b}$ are said to be parallel if there exists a scalar k such that $\vec{a} = k\vec{b}$
- **Like Vectors :**

  Two vectors are said to be like vectors if they are parallel and have same direction.
- **Unlike Vectors :**

  Two vectors are said to be unlike vectors if they are parallel and have opposite direction.
- **Collinear Vectors :**

  Two vectors are said to be co-linear if they lie on one line.
- **Co-initial Vectors :**

  Vectors having the same initial point are called co-initial vectors.
- **Coplanar Vectors :**

  Vectors are said to be coplanar if they lie on one plane.
- **Negative of a Vector :**

  A vector which has the same magnitude as the vector $\vec{a}$ but in opposite direction is called the negative of $\vec{a}$. It is denoted by $-\vec{a}$.
- **Position vector of a point :**

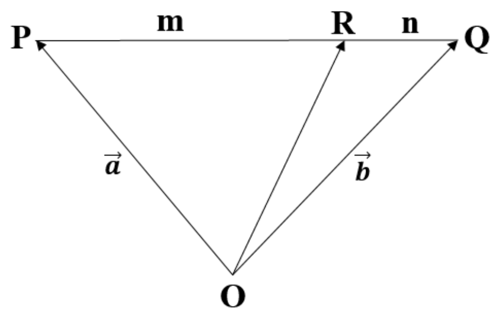
  If O be a fixed point and P be any other point, then the vector $\overrightarrow{OP}$ is defined as the position vector of P with respect to O.
- If P and Q are two points whose position vectors are $\vec{a}$ and $\vec{b}$ respectively then

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \vec{b} - \vec{a}$$

- **Note :**

  If P and Q are two points whose position vectors are $\vec{a}$ and $\vec{b}$ respectively, then the position vector of a point R which divides the line segment joining P and Q internally in the ratio $m : n$ given by

$$\overrightarrow{OR} = \frac{m\vec{b} + n\vec{a}}{m + n}$$



- If R is the middle point of PQ then the position vector of the point R is given by

$$\overrightarrow{OR} = \frac{\vec{a} + \vec{b}}{2}$$

Note :

- Position vector a point $P(x, y)$ with respect to origin in coordinate plane is given by $\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath}$ where $\hat{\imath}$ and $\hat{\jmath}$ are unit vectors along X- axis, Y- axis respectively.
- Position vector of a point $P(x, y, z)$ with respect to origin in space is given by $\overrightarrow{OP} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$
  where $\hat{\imath}, \hat{\jmath}$ and $\hat{k}$ are unit vectors along X- axis, Y- axis and Z - axis respectively.
- If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points, then

$$\overrightarrow{PQ} = (x_2 - x_1)\hat{\imath} + (y_2 - y_1)\hat{\jmath} + (z_2 - z_1)\hat{k}$$

- **Magnitude and Direction of a Vector :**
  Let $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ be a vector. Then magnitude of $\vec{a}$ is defined by

$$|\vec{a}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

- D.c.s of $\vec{a}$ is given by

$$cos\alpha = \frac{a_1}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, cos\beta = \frac{a_2}{\sqrt{a_1^2 + a_2^2 + a_3^2}}, cos\gamma = \frac{a_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}}$$

- Unit vector along a vector $\vec{a}$ is denoted by $\hat{a}$ and $\hat{a} = \frac{\vec{a}}{|\vec{a}|}$

Algebra of Vectors :

- **Addition :**

$$[a_1, a_2, a_3] + [b_1, b_2, b_3] = [a_1 + b_1, a_2 + b_2, a_3 + b_3]$$

- **Subtraction :**

$$[a_1, a_2, a_3] - [b_1, b_2, b_3] = [a_1 - b_1, a_2 - b_2, a_3 - b_3]$$

- **Multiplication**

  Scalar Multiplication :

$$K[a_1, a_2, a_3] = [Ka_1, Ka_2, Ka_3]$$

MODULE - 17

SCALAR AND VECTOR PRODUCT

Scalar or Dot Product :

- The scalar product of the vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a}.\vec{b}$ and defined as $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|cos\theta$ where θ is the angle between $\vec{a}$ and $\vec{b}$.
- $\vec{a}$ is perpendicular to $\vec{b}$ iff $\vec{a}.\vec{b} = 0$
- $\hat{\imath}.\hat{\imath} = 1 = \hat{\jmath}.\hat{\jmath} = \hat{k}.\hat{k}$
- $\hat{\imath}.\hat{\jmath} = 0 = \hat{\jmath}.\hat{k} = \hat{k}.\hat{\imath}$

Properties of Dot Product :

- Dot product is commutative. i.e. $\vec{a}.\vec{b} = \vec{b}.\vec{a}$
- Dot product is distributive over vector addition. i.e. $\vec{a}.\left(\vec{b} + \vec{c}\right) = \vec{a}.\vec{b} + \vec{a}.\vec{c}$
- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then $\vec{a}.\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$
- Scalar projection of $\vec{a}$ on $\vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}$
- Vector projection of $\vec{a}$ on $\vec{b} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\hat{b}$
- Scalar projection of $\vec{b}$ on $\vec{a} = \frac{\vec{a}.\vec{b}}{|\vec{a}|}$
- Vector projection of $\vec{b}$ on $\vec{a} = \left(\frac{\vec{a}.\vec{b}}{|\vec{a}|}\right)\hat{a}$

Component of a vector along and perpendicular to a vector :

- Component of $\vec{a}$ along $\vec{b} = \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\hat{b}$
- Component of $\vec{a}$ perpendicular to $\vec{b} = \vec{a} - \left(\frac{\vec{a}.\vec{b}}{|\vec{b}|}\right)\hat{b}$
- Component of $\vec{b}$ along $\vec{a} = \left(\frac{\vec{a}.\vec{b}}{|\vec{a}|}\right)\hat{a}$
- Component of $\vec{b}$ perpendicular to $\vec{a} = \vec{b} - \left(\frac{\vec{a}.\vec{b}}{|\vec{a}|}\right)\hat{a}$
- $\vec{a}^2 = \vec{a}.\vec{a} = |\vec{a}|^2$

Vector Product or Cross Product :

- The vector product of the vectors $\vec{a}$ and $\vec{b}$ is denoted by $\vec{a} \times \vec{b}$ and is defined as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| sin\theta \hat{n}$$

where θ is the angle between $\vec{a}, \vec{b}$ and $\hat{n}$ is the unit vector perpendicular to the plane containing $\vec{a}$ and $\vec{b}$.

- $\vec{a}$ is parallel to $\vec{b}$ $\quad iff \quad \vec{a} \times \vec{b} = \vec{0}$
- $\hat{\imath} \times \hat{\imath} = \vec{0} = \hat{\jmath} \times \hat{\jmath} = \hat{k} \times \hat{k}$
- $\hat{\imath} \times \hat{\jmath} = \hat{k},\ \hat{\jmath} \times \hat{k} = \hat{\imath},\ \hat{k} \times \hat{\imath} = \hat{\jmath}$
- $\hat{\jmath} \times \hat{\imath} = -\hat{k},\ \hat{k} \times \hat{\jmath} = -\hat{\imath},\ \hat{\imath} \times \hat{k} = -\hat{\jmath}$
- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$ and $\vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ then $\vec{a} \times \vec{b} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

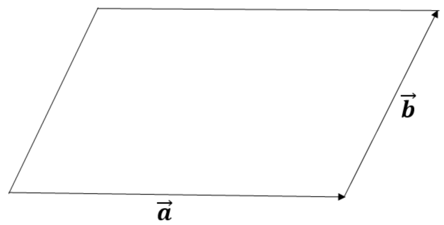
Properties of Cross Product :

(i) Cross product is not commutative. i.e. $\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

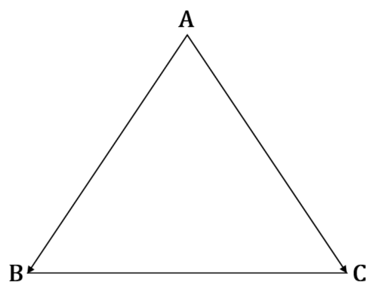
(ii) Cross product is distributive over vector addition.

i.e. $\vec{a} \times (\vec{b} + \vec{c}) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

- Area of a parallelogram whose adjacent sides are $\vec{a}$ and $\vec{b} = |\vec{a} \times \vec{b}|$



- Area of a triangle ABC $= \frac{1}{2}|\overrightarrow{AB} \times \overrightarrow{AC}|$



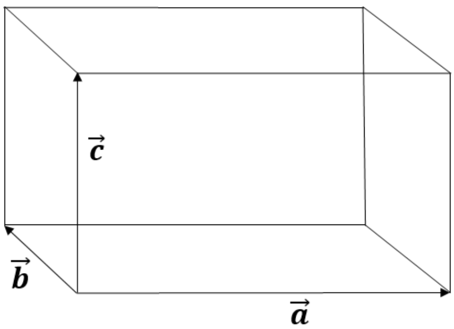
- The unit vector perpendicular to the vectors $\vec{a}$ and $\vec{b} = \frac{\vec{a} \times \vec{b}}{|\vec{a} \times \vec{b}|}$

Scalar Triple Product :

- The scalar triple product of $[\vec{a}\ \vec{b}\ \vec{c}] = \vec{a}.(\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}).\vec{c}$
- $\vec{a}, \vec{b}, \vec{c}$ are coplanar iff $[\vec{a}\ \vec{b}\ \vec{c}] = 0$
- If $\vec{a} = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k},\ \vec{b} = b_1\hat{\imath} + b_2\hat{\jmath} + b_3\hat{k}$ and $\vec{c} = c_1\hat{\imath} + c_2\hat{\jmath} + c_3\hat{k}$ then

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

- Volume of a parallelepiped whose edges are $\vec{a}, \vec{b}, \vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]$



- Volume of a tetrahedron whose sides are $\vec{a}, \vec{b}, \vec{c} = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$

Properties :

(i) If any two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are equal, then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(ii) If any two of the three vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel, then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$

(iii) $[\vec{a}\ \vec{b}\ \vec{c} + \vec{d}] = [\vec{a}\ \vec{b}\ \vec{c}] + [\vec{a}\ \vec{b}\ \vec{d}]$

(iv) $[\vec{a}\ k\vec{b}\ \vec{c}] = k[\vec{a}\ \vec{b}\ \vec{c}]$

Vector Triple Product :

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$$

The vector equation of a straight line :

(i) The vector equation of a straight line passes through a point with position vector $\vec{a}$ and parallel to a vector $\vec{b}$ is $\vec{r} = \vec{a} + t\vec{b}$, where t is a parameter.

(ii) The vector equation of a straight line passes through two points with position vectors $\vec{a}$ and $\vec{b}$ is $\vec{r} = \vec{a} + t(\vec{b} - \vec{a})$

(iii) Vector equation of a straight line passes through a point with position vector $\vec{a}$ and perpendicular to two non-parallel vectors $\vec{b}$ and $\vec{c}$ is $\vec{r} = \vec{a} + t(\vec{b} \times \vec{a})$

The Vector Equation of a Plane :

(i) The vector equation of a plane passes through a point Q with position vector $\vec{a}$ and perpendicular to $\vec{n}$ is $(\vec{r}-\vec{a}).\vec{n}=0$ where $\vec{r}$ is the position vector of any point P on the plane.

(ii) The vector equation of a plane passes through a given point Q with position vector $\vec{a}$ and parallel to vectors $\vec{b}$ and $\vec{c}$ is $\vec{r}=\vec{a}+t\vec{b}+s\vec{c}$, where t and s are parameters.

(iii) Vector equation of a plane passing through the points P and Q with position vectors $\vec{a}\ and\ \vec{b}$ respectively and parallel to $\vec{c}$ is

$$\vec{r}=(1-t)\vec{a}+t\vec{b}+s\vec{c}$$

where t and s are parameters.

(iv) Vector equation of a plane passes through three non collinear points P, Q and R with position vectors $\vec{a},\vec{b}\ and\ \vec{c}$ respectively is

$$\vec{r}=(1-s-t)\vec{a}+t\vec{b}+s\vec{c}$$

where t and s are parameters.

***