Q&A

WORK BOOK
CUM
QUESTION BANK WITH ANSWERS

MATHEMATICS

CLASS - XII

SCHEDULED CASTES & SCHEDULED TRIBES
RESEARCH & TRAINING INSTITUTE (SCSTRTI)
ST & SC DEVELOPMENT DEPARTMENT
BHUBANESWAR

Work Book
cum
Question Bank with Answers

MATHEMATICS

CLASS-XII

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2020

FOREWORD

An innovative education program has been initiated by ST & SC Development Department, Govt. of Odisha for the students appearing in +2 Science and Commerce examination pursuing studies in the ST & SC Development Department Schools (EMRS & HSS) to ensure quality education at +2 level.

In this regard it is to mention that an Academic Performance Monitoring Cell (APMC)has been set up in SCSTRTI to monitor the Training and Capacity Building of Teachers of SSD Higher Secondary Schools and Ekalabya Model Residential Schools (EMRS) to enhance quality education for better performance of the students appearing +2 Science and Commerce examination.This effort by APMC will certainly help the students to equip themselves for appropriate answering the question in the examination in an efficient manner.

In order to materialize the effort, thebest ofsubject experts of the state have been roped into formulate self-contained and self-explanatory "Work book cumQuestions Bankwith Answers" as per the syllabi of CHSE,Odisha.They have tried to make the material as far as activity based and solution based as possible.This novel effort is first of its kind at +2 level in Odisha.

I would like to extend my thanks to Prof.(Dr.) A.B. Ota, Advisor-Cum-Director and Special Secretary, SCSTRTI and the team of Subject experts for their sincere effort for bringingout the study materials in quick time.

Hope, these study materials will be extremely useful for the students appearing the +2 examination in Science and Commerce of our SSD Schools.

Ranjana Chopra
Principal Secretary
ST & SC Development Department
Govt. of Odisha

PREFACE

The ST and SC Development Department, Government of Odisha, has initiated an innovative effort by setting up an Academic Performance Monitoring Cell (APMC) in Scheduled Castes and Scheduled Tribes Research and Training Institute (SCSTRTI) to monitor the Training and Capacity Building of teachers of SSD Higher Secondary Schools and Ekalavya Model Residential Schools (EMRS) and to ensure quality education of students studying at +2 level under the administrative control of the ST & SC Development Department. This innovative programme is intended to ensure quality education in the Higher Secondary Level of the schools of the ST & SC Development Department.

Since the introduction of +2 Science and +2 Commerce stream by the Council of Higher Secondary Education, Odisha, there was a great demand to cater to the needs of the students appearing the +2 Examination. But no organisation or institute has taken the initiative to fulfil the needs of the students appearing the +2 examination. Realizing the necessities and requirements of students to perform better and secure better marks in the examination and proper pattern of answering the question in a scientific way, the APMC under the banner of SCSTRTI has taken the initiative for the first time in Odisha to prepare Questions Banks in Physics, Chemistry, Botany, Zoology, Mathematics, IT, English & Odia of the Science Stream and all the disciplines of the Commerce stream in line with the Syllabus of the Council of Higher Secondary Education (CHSE).

These questions banks are first of this kind in Odisha, as per syllabi of CHSE and are self contained and self explanatory. The subject expert, who are the best in their respective subjects in the state have been roped in for the exercise. They have given their precious time to make the question banks as activity based and solution based as possible.

I take this opportunity to thank all the subject experts of different subjects for rendering help and assistance to prepare the question banks within a record time. I hope, this material will be extremely useful for the students preparing for the +2 examination in different subjects of Science & Commerce streams.

Prof. (Dr.) A.B. Ota
Advisor cum Director & Special Secretary
SCSTRTI, Govt. of Odisha

MATHEMATICS (2nd Year) Syllabus

General Imstructions :

1. All questions are compulsory in Group A, which are very short answer type questions. All questions in the group are to be answered in one word, one sentences or as per exact requirement of the question. (1x10=10 Marks)
2. Group-B contain 5(five) questions and each question have 5 bits, out of which only 3 bits are to be answered (Each bit caries 4 Marks) (4 x15=60 Marks)
3. Group-C contains 5(five) questions and each question contains 2/3 bits, out of which only 1(one) bit is to be answered. Each bit caries 6 (six) Mark (6x5 =30 Marks)

UNIT - I : Relations and Functions

1. **Relations and Functions**

Types of relations ; reflexive, symmetric, transitive and equivalence relations. One to one and onto functions, composite functions, inverse of function. Binary operations.

2. **Inverse Trigonometric Functions**

Definition, range, domain, principle value branch. Graphs of inverse trigonometric functions. Elementary properties of inverse trigonometric functions.

3. **Linear Programming**

Introduction, related terminology such as constraints, objective function, optimization, different types of linear programming (L.P.) problems, mathematical formulation of L.P. problems, graphical method of solution for problems in two variables, feasible and infeasible regions (bounded and unbounded), feasible and infeasible solutions, optimal feasible solutions (up to three non-trivial constraints).

UNIT - II : Algebra

1. **Matrices**

Concept, notation, order, equality, types of matrices, zero and identity matrix, transpose of a matrix, symmetric and skew symmetric matrices. Operation on matrices; Addition and multiplication and multiplication with a scalar. Simple properties of addition, multiplication and scalar multiplication. Non commutativity of multiplication of matrices and existence of non-zero matrices whose product is the zero matrix (restrict to square matrices of order 2). concept of elementary row and column operations. Invertible matrices and proof of the uniqueness of inverse, if it exists; (Here all matrices will have real entries).

2. **Determinants**

Determinant of a square matrix (up to 3 × 3 matrices), properties of determinants, minors, co- factors and applications of determinants in finding the area of a triangle, Adjoint and inverse of a square matrix. Consistency, inconsistency and number of solutions of system of linear equations by examples, solving system of linear equations in two or three variables (having unique solution) using inverse of a matrix.

3. **Probability**

Conditional probability, multiplication theorem on probability. Independent events, total probability, Baye's theorem, Random variable and its probability distribution, mean and variance of random variable. Independent (Bernoulli) trials and Binomial distribution.

UNIT-III : Differential Calculus

1. **Continuity and Differentiability**

Continuity and differentiability, derivative of composite functions, chain rule, derivatives of inverse trigonometric functions, derivative of implicit functions. Concept of exponential and logarithmic functions.

Derivatives of logarithmic and exponential functions. Logarithmic differentiation, derivative of functions expressed in parametric forms. Second order derivatives. Rolle's and Lagrange's Mean Value Theorems (without proof) and their geometric interpretation.

2. **Applications of Derivatives**

Applications of derivatives : rate of change of bodies, increasing and decreasing functions, tangents and normals, use of derivatives in approximation, maxima and minima (first derivative test motivate geometrically and second derivative test given as a provable tool). Simple problems (that illustrate basic principles and understanding of the subject as well as real-life situations).

UNIT-IV Integral Calculus

1. **Integrals**

Integration as inverse process of differentiation. Integration of a variety of functions by substitution, by partial fractions and by parts, Evaluation of simple integrals of the following types and problems based on them.

$$\int \frac{dx}{x^2 \pm a^2}, \int \frac{dx}{a^2 - x^2}, \int \frac{dx}{ax^2 + bx + c}$$

$$\int \frac{dx}{ax^2 + bx + c}, \int \frac{px + q}{ax^2 + bx + c} dx,$$

$$\int \frac{px + q}{ax^2 + bx + c} dx, \int \sqrt{a^2 \pm x^2} dx,$$

$$\int \sqrt{x^2 - a^2} dx,$$

$$\int \sqrt{ax^2 + bx + ca^2} dx, \int (px + q)\sqrt{ax^2 + bx + c} \; dx$$

Definite integrals as a limit of a sum, Fundamental Theorem of Calculus (without proof). Basic properties of definite integrals and evaluation of definite integrals.

2. **Applications of the Integrals**

Applications in finding the area under simple curves, especially lines, circles/parabolas/ellipses (in standard form only). Area between any of the two above said curves (the region should be clearly identifiable).

3. **Differential Equations.**

Definition, order and degree, general and particular solutions of a differential equation. Formation of differential equation whose general solution is given. Solution of differential equations by method of separation of variables, solutions of homogeneous differential equations of first order and first degree. Solutions of linear differential equation of the type :

$\frac{dy}{dx} + py = q,$ where p and q are functions of x or constants.

$\frac{dx}{dy} + px = q,$ where p and q are functions of y or constants.

UNIT - V : Vectors and Three-Dimensional Geometry

1. **Vectors**

Vectors and scalars, magnitude and direction of a vector. Direction cosines and direction ratios of a vector. Types of vectors (equal, unit, zero, parallel and collinear vectors), position vector of a point, negative of a vector, components of a vector, addition of vectors, multiplication of a vector by a scalar, position vector of a point dividing a line segment in a given ratio. Definition, Geometrical Interpretation, properties and application of scalar (dot) product of vectors, vector (cross) product of vectors, scalar triple product of vectors, Coplanarity of three vectors.

2. **Three - dimensional Geometry**

Direction cosines and direction ratios of a line joining two points. Cartesian equation and vector equation of a line, coplanar and skew lines, shortest distance between two lines. Cartesian and vector equation of a plane. Angle between (i) two lines, (ii) two planes, (iii) a line and a plane. Distance of a point from a plane.

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QUESTION PATTERN OF CHSE

Unit	Topic	Marks	No. of Periods
I	Relations and Functions & Linear Programming	20	45
II	Algebra and Probability	20	45
III	Differential Calculus	20	45
IV	Integral Calculus	20	45
V	Vector 3-D Geometry	20	45
	Total	**100**	**220**

CHSE QUESTION PAPERS WITH ANSWERS

2019 to 2017

2019 (A)

Full Marks : 100 Time : 3 hours

The figures in the right-hand margin indicate marks.
Answer the questions of all the Groups as directed.
Electronic gadgets are not allowed in the Examination Hall

GROUP - A
(Marks : 10)

1. Answer all questions : [1x10=10]

(a) If $\phi(x) = f(x) + f(1-x)$, $f''(x) = 0$ for $0 \le x \le 1$, then is $x = \frac{1}{2}$ a point of maxima or minima of $\phi(x)$?

(b) If f is an odd function, then write the value of $\int_{-a}^{a} \frac{f(\sin x)}{f(\cos x) + f(\sin^2 x)} dx$

(c) Write the order of the differential equation whose solution is given by
$y = (c_1 + c_2)\cos(x + c_3) + c_4 e^{x+c_5}$
where c_1, c_2, c_3, c_4 and c_5 are arbitrary constants.

(d) If $\vec{a} = \vec{b} + \vec{c}$, then write the value of $\vec{a}.(\vec{b} \times \vec{c})$. .

(e) Write the value of k such that the line $\frac{x-4}{1} = \frac{y-2}{1} = \frac{z-k}{2}$ lies on the plane $2x - 4y + z = 7$.

(f) If R is relation on A such that $R = R^{-1}$, then write the type of the relation R.

(g) Write the value of $\cos^{-1}\cos(3\pi/2)$.

(h) If $\begin{vmatrix} 1+x & x & x^2 \\ x & 1+x & x^2 \\ x^2 & x & 1+x \end{vmatrix}$
$= a + bx + cx^2 + dx^3 + ex^4 + fx^5$
then write the value of a.

(i) Let A and B be two mutually exclusive events such that $P(A) = \frac{1}{2}$ and $P(B) = \frac{1}{3}$. Write the value of $P(A \cap B)$.

(j) If $f'(2^+) = 0$ and $f'(2^-) = 0$, then is f(x) continuous at x = 2 ?

GROUP - B
(Marks : 60)

2. Answer any three questions : [4x3=12]

(a) Prove that
$$\cos^{-1}\left(\frac{b + a\cos x}{a + b\cos x}\right) = 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

(b) Two types of food X and Y are mixed to prepare a mixture in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. These vitamins are available in 1 kg of food as per the table given below :

	Vitamin		
Food	A	B	C
X	1	2	3
Y	2	2	1

1 kg of food X costs ₹ 16 and 1 kg of food Y costs ₹ 20. Formulate the LPP so as to determine the least cost of the mixture containing the required amount of vitamins.

(c) Construct the multiplication table X_7 on the set {1, 2, 3, 4, 5, 6}. Also find the inverse element of 4 if it exist.

(d) Let R be a relation on the set R of real numbers such that $\alpha R b$ if a - b is an integer. Test whether R is an equivalence relation. If so, find the equivalence class of 1 and $\frac{1}{2}$.

(e) Solve : $2\tan^{-1}(\cos x) = \tan^{-1}(2\operatorname{cosec} x)$

3. Answer any three questions : [4x3=12]

(a) Find the probability distribution of number of heads in 3 tosses of a fair coin.

(b) If $A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -2 & 5 & 3 \end{bmatrix}$, then verify that $A + A'$ is symmetric and $A - A'$ is skew-symmetric.

(c) If $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$ then show that $A^3 - 23A - 40I = 0$.

(d) Solve : $\begin{bmatrix} x+1 & \omega & \omega^2 \\ \omega & x+\omega^2 & 1 \\ \omega^2 & 1 & x+\omega \end{bmatrix} = 0$

(e) A person takes 4 tests in succession. The probability of his passing the first test is p, that of his passing each succeeding test is p or $\frac{p}{2}$, depending on his passing or failing the preceding test. Find the probability of his passing just 3 tests.

4. Answer any three questions : [4x3=12]

(a) Find the point on the curve $x^2 + y^2 - 4xy + 2 = 0$, where the normal to the curve is parallel to the x-axis.

(b) Find the intervals in which the function $y = \frac{1n\,x}{x}$ is increasing and decreasing.

(c) If $y = e^{x^{e^{x^{e^{x\ldots}}}}}$, then find $\frac{dy}{dx}$.

(d) Find $\frac{d^2y}{dx^2}$, if $x = a\cos\theta$ and $y = b\sin\theta$.

(e) Verify Lagrange's mean value theorem for $f(x) = x^3 - 2x^2 - x + 3$ on [1, 2].

5. Answer any three questions : [4x3=12]

(a) Find the differential equation of the curve $y = ae^{3x} + be^{5x}$.

(b) Solve :

$$(x^2 + 7x + 12)dy + (y^2 - 6y + 5)dx = 0$$

(c) Evaluate : $\int \frac{2x+1}{\sqrt{x^2 + 10x + 29}} dx$

(d) Evaluate : $\int_0^{\pi/2} \frac{\cos x\, dx}{(2 - \sin x)(3 + \sin x)}$

(e) Find the area of the region bounded by the curve $y = 6x - x^2$ and the x-axis.

6. Answer any three questions : [4x3=12]

(a) If $\langle l_1, m_1, n_1 \rangle$ and $\langle l_2, m_2, n_2 \rangle$ are direction cosines of two mutually perpendicular lines, then show that the direction cosines of the line perpendicular to both of them are

$$\langle m_1n_2 - m_2n_1, n_1l_2 - n_2l_1, l_1m_2 - l_2m_1 \rangle.$$

(b) Find the point where the line $\frac{x-2}{1} = \frac{y}{-1} = \frac{z-1}{2}$ meets the plane $2x + y + z = 2$.

(c) Find a unit vector perpendicular to each of the vectors $\vec{a} + \vec{b}$ and $\vec{a} - \vec{b}$, where $\vec{a} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$.

(d) Show that $(\vec{a} \times \vec{b})^2 = a^2b^2 - (\vec{a}.\vec{b})^2$.

(e) Find the vector equation of a plane which is at a distance of 3 units from the origin, $2\hat{i} + 3\hat{j} - 6\hat{k}$ being a normal to the plane. Also get its Cartesian equation.

GROUP - C

(Marks : 30)

7. Answer any one question : [6]

(a) If $e^{y/x} = \frac{x}{a+bx}$, then show that

$$x^3 \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(x\frac{dy}{dx} - y\right)^2$$

(b) Show that the shortest distance of the point (0, 8a) from the curve $ax^2 = y^3$ is $2a\sqrt{11}$.

8. Answer any one question : [6]

(a) Determine the area common to the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2x$.

(b) Solve : $y^2 + x^2 \frac{dy}{dx} = xy\frac{dy}{dx}$

(c) Evaluate : $\int \frac{dx}{2\cos^2 x + 3\cos x}$

9. Answer any one question : [6]

(a) Show by vector method that the four points (6,2,-1), (2,-1,3), (-1,2,-4) and (-12,-1,-3) are coplanar.

(b) Find the distance of the point (1, -1, -10) from the line $\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ measured parallel to the line $\frac{x+2}{2} = \frac{y-3}{-3} = \frac{z-4}{8}$.

10. Answer any one question : [6]

(a) If $\sin^{-1}\left(\frac{x}{a}\right) + \sin^{-1}\left(\frac{y}{b}\right) = \sin^{-1}\left(\frac{c^2}{ab}\right)$

then prove that

$$b^2x^2 + 2xy\sqrt{a^2b^2 - c^4} + a^2y^2 = c^4.$$

(b) Solve the following LPP graphically :

Maximize $Z = 10x_1 + 12x_2 + 8x_3$ subject to

$$x_1 + 2x_2 \le 30$$
$$5x_1 - 7x_3 \ge 12$$
$$x_1 + x_2 + x_3 = 20$$
$$x_1, x_2, x_3 \ge 0$$

(c) Prove that $f : X \to Y$ is injective iff for all subsects A, B of X, $f(A \cap B) = f(A) \cap f(B)$.

11. Answer any one question : [6]

(a) Examining consistency and solvability, solve the following equations by matrix method :

$$x - 2y = 3$$
$$3x + 4y - z = -2$$
$$5x - 3z = -1$$

(b) Out of the adult population in a village, 50% are farmers, 30% do business and 20% are service holders. It is known that 10% of farmers, 20% of businessmen and 50% of service holders are above poverty line. What is the probability that a villager chosen from the adult population of the village, selected at random, is above poverty line ?

(c) Find the inverse of the following matrix using elementary transformation :

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$$

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ANSWERS

2019 (A) - ANSWERS

GROUP - A

1. Answer all questions : [1x10=10]

(a) $x=\frac{1}{2}$ is not a point of maxima or minima of $\phi(x)$.

Reason :

$$\phi(x)=f(x)+f(1-x)$$

$$\therefore \phi'(x)=f'(x)-f'(1-x)$$

$$\phi''(x)=f''(x)+[f''(1-x)]$$

$$=f''(x)+f''(1-x)$$

$$\phi''\left(\frac{1}{2}\right)=f''\left(\frac{1}{2}\right)+f''\left(1-\frac{1}{2}\right)$$

$$=f''\left(\frac{1}{2}\right)+f''\left(\frac{1}{2}\right)$$

$$=2f''\left(\frac{1}{2}\right)$$

$$=2.0 \qquad [\because f''(x)=0 \text{ for } 0\le x\le 1]$$

$$=0$$

Since second derivative of $\phi''(x)=0$ so $\phi(x)$ is neither maximum nor minimum at $x=\frac{1}{2}$.

(b) $\int_{-a}^{a}\frac{f(\sin x)}{f(\cos x)+f(\sin^2 x)}=0$

Reason

Since f is an odd function, f(-x) = -f(x).

Let $f(x)=\frac{f(\sin x)}{f(\cos x)+f(\sin^2 x)}$

$$f(-x)=\frac{f[\sin(-x)]}{f[\cos(-x)]+f[\sin^2(-x)]}$$

$$=\frac{f[-\sin x]}{f(\cos x)+f(\sin^2 x)}$$

$$=\frac{-f[\sin x]}{f(\cos x)+f(\sin^2 x)}$$

[$\because$ f is an odd function]

$$=-\left[\frac{f(\sin x)}{f(\cos x)+f(\sin^2 x)}\right]$$

$$=-f(x)$$

So f(x) is an odd function.

$$\therefore \int_{-a}^{a}f(x)=0$$

$$\Rightarrow \int_{-a}^{a}\frac{f(\sin x)}{f(\cos x)+f(\sin^2 x)}dx=0$$

(c) Order of the differential equation whose solution is

$$y=(C_1+C_2)\cos(x+C_3)+C_4e^{x+C_5}$$

is three as there are three distinct constant in the solution.

Reason

Solution is

$$y=(C_1+C_2)\cos(x+C_3)+C_4e^{x+C_5}$$

$$=A\cos(x+C_3)+C^4e^x e^{C_5}$$

where $A=C_1+C_2$

$$=A\cos(x+C_3)+Be^x \text{ where } B=C_4e^{C_5}$$

Three distinct constants are A, B, C_3.

(d) $\vec{a}.(\vec{b}\ x\ \vec{c})=0$

Reason

Given $\vec{a}=\vec{b}+\vec{c}$

$$\therefore \vec{a}.(\vec{b}\,x\,\vec{c})=(\vec{b}+\vec{c}).(\vec{b}\,x\,\vec{c})$$

$$=\vec{b}.(\vec{b}\,x\,\vec{c})+\vec{c}.(\vec{b}\,x\,\vec{c})$$

$$=0+0=0$$

[A scalar triple product is zero if any two vectors are equal]

(e) K = 7

Reason

Given line is $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ ...(1)

The given plain is $2x-4y+z=7$ ...(2)

A point on the line is (4, 2, k)

This point will lie on the plane (2)

if 2.4 - 4.2 + k = 7

$\Rightarrow$ k = 7.

(f) If $R=R^{-1}$ then the relation R is symmetric.

(g) $\frac{\pi}{2}$

Reason

$$\text{Cos}^{-1}\text{ Cos}\left(\frac{3\pi}{2}\right)=\text{Cos}^{-1}0=\frac{\pi}{2}$$

(h) If $\begin{vmatrix}1+x & x & x^2\\ x & 1+x & x^2\\ x^2 & x & 1+x\end{vmatrix}$

$=a+bx+cx^2+dx^3+ex^4+fx^5$ then a=1.

(i) 0

Reason

Since A and B are mutually exclusive events, $A\cap B=\phi$

$$P(A\cap B)=\frac{|A\cap B|}{|S|}=\frac{|\phi|}{|S|}=\frac{0}{|S|}=0$$

(j) f(x) is continuous at x = 2 ?

Reasons

$$f'(2^+)=f'(2^-)=0$$

$\Rightarrow$ The function f(x) is differentiable at x=2. Since every derivable function is continuous at x=2, so f(x) is continuous at x = 2.

GROUP - B

2. (a) Let $\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)=\theta$

$$\Rightarrow \tan\theta=\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

$$\text{Cos}2\theta=\frac{1-\tan^2\theta}{1+\tan^2\theta}$$

$$=\frac{1-\frac{a-b}{a+b},\tan^2\frac{x}{2}}{1+\frac{a-b}{a+b}\tan^2\frac{x}{2}}$$

$$=\frac{(a+b)-(a-b)\tan^2\frac{x}{2}}{(a+b)+(a-b)\tan^2\frac{x}{2}}$$

$$=\frac{a\left(1-\tan^2\frac{x}{2}\right)+b\left(1+\tan^2\frac{x}{2}\right)}{a\left(1+\tan^2\frac{x}{2}\right)+b\left(1-\tan^2\frac{x}{2}\right)}$$

$$=\frac{a\left(1-\tan^2\frac{x}{2}\right)+b}{a+b\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)}$$

$$=\frac{a\cos x+b}{a+b\cos x}$$

$$\Rightarrow 2\theta=\cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

$$\Rightarrow 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$$

$$=\cos^{-1}\left(\frac{a\cos x+b}{a+b\cos x}\right)$$

(b) Let x kg of food X and y kg of food Y are required to prepare a mixture at least cost.

Given that

1 kg of food X costs Rs. 16.00

$\therefore$ x kg of food X costs Rs. 16x

Also 1 kg of food Y costs Rs. 20.00

y kg of food Y costs Rs. 20y

Total cost is Z = 16x + 20y

Total units of Vitamin A = x + 2y

Total units of Vitamin B = 2x + 2y

Total units of Vitamin C = 3x + y

The formulation of the linear programming problem is as follows :

Minimize Z = 15x + 20y

Subject to $x + 2y \geq 20$

$2x + 2y \geq 12$

$3x + y \geq 8$

and $x, y \geq 0$

(c) The given set is {1, 2, 3, 4, 5, 6}

The multiplication table X_7 is as follows

X_7	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	3
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

We see that all the elements in the composition table are the elements of the set, the inverse element of 4 is 2 as 4x2=1

(d) The set of real numbers is R.

For a, b $\in$ R, the relation R on the set R is defined by

aRb $\Rightarrow$ a - b is an integer.

$\Rightarrow$ a - b = k where k is an integer.

We shall test whether R is as equivalence relation.

Reflexive

For all a $\in$ R

a - a = 0

$\Rightarrow$ a - a is an integer

$\Rightarrow$ aRa is true

$\Rightarrow$ R is reflexive

Symmetric

For a, b $\in$ R

aRb $\Rightarrow$ a - b is an integer

$\Rightarrow$ a - b = k where k is an integer

$\Rightarrow$ b - a = - k where - k is an integer

$\Rightarrow$ bRa

So the relation R is symmetric.

Transetive

For a, b, c $\in$ R

aRb $\Rightarrow$ a - b is an integer

$\Rightarrow$ a - b= k where k is an integer...(1)

bRc $\Rightarrow$ b - c is an integer

$\Rightarrow$ b-c= k, where k, is an integer...(2)

Adding (1) and (2), we get

(a - b) + (b - c) = k + k_1

$\Rightarrow$ a- c = k + k_1 (k+k_1 is an integer)

$\Rightarrow$ a - c is an integer

$\Rightarrow$ aRc.

$\therefore$ aRb, bRc $\Rightarrow$ aRc

So R is transitive.

Since R is reflexive, symmetric and transitive, it is an equivalence relation.

R_1= [1] = Equivalence class of 1

= $\{x \in R : 1Rx\}$

= $\{x \in R : 1 - x$ is an integer$\}$

= {... -2, -1, 0, 1, 2, 3, 4,5, ...}

= The set of integer.

$R_{\frac{1}{2}} = \left(\frac{1}{2}\right)$ = Equivalence class of $\frac{1}{2}$

= $\{x \in R : \frac{1}{2} Rx\}$

= $\{x \in R : \frac{1}{2} x$ is an integer$\}$

= $\{... -\frac{3}{2}, -\frac{1}{2}, -\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, ...\}$

(e) $2\tan^{-1}(\cos x) = \tan^{-1}(2\,\text{cosec}\,x)$

$\Rightarrow \tan^{-1}\dfrac{2\cos x}{1-\cos^2 x} = \tan^{-1}(2\,\text{cosec}\,x)$

$\Rightarrow \tan^{-1}\dfrac{2\cos x}{\sin^2 x} = \tan^{-1}(2\,\text{cosec}\,x)$

$\Rightarrow \tan^{-1}(2\,\text{cosec}\,x\cot x) = \tan^{-1}(2\,\text{cosec}\,x)$

$\Rightarrow 2\,\text{cosec}\,x\cot x = 2\,\text{cosec}\,x$

$\Rightarrow 2\,\text{cosec}\,x\cot x - 2\,\text{cosec}\,x = 0$

$\because \text{cosec}\,x \neq 0$

$\therefore \cot x - 1 = 0$

$\Rightarrow \cot x = 1 = \cot\dfrac{\pi}{4}$

$\therefore x = n\pi + \dfrac{\pi}{4}$, where $n \in Z$.

3. (a) A fair can is tossed three times.

Let x be the random variable of getting heads. The range of X is {0, 1, 2, 3}

P (x = 0) = Probability of getting no head

$= P(TTT) = \dfrac{1}{8}$

P(x=1) = Probability of getting one head

= P(HTT, THT, TTH)

$= \dfrac{3}{8}$

P(x=2) = Probability of getting two heads

$= P(HHT, HTH, THH) = \dfrac{3}{8}$

P(x=3) = Probability of getting three heads

$= P(HHH) = \dfrac{1}{8}$.

The values of x and the corresponding probabilities can be exhibited as :

X :	0	1	2	3
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

Alternative Method

A fair coin is tossed three three times. Let x be the random variable of getting heads.

x takes the values 0, 1, 2, 3,

In one toss, $P(H) = \dfrac{1}{2}$, $P(T) = \dfrac{1}{2}$

There are three tosses.

This experiment is a binomial distribution with $n = 3$, $p = \dfrac{1}{2}$, $q = \dfrac{1}{2}$.

$\therefore P(x=0) = 3_{C_0}p^0q^3 = 1.1.\left(\dfrac{1}{2}\right)^3 = \dfrac{1}{8}$

$\therefore P(x=1) = 3_{C_1}p^1q^2 = 3.\dfrac{1}{2}.\left(\dfrac{1}{2}\right)^2 = \dfrac{3}{8}$

$P(x=2) = 3_{C_2}p^2q^1 = 3.\left(\dfrac{1}{2}\right)^2.\dfrac{1}{2} = \dfrac{3}{8}$

$P(x=3) = 3_{C_3}p^3q^0 = 1.\left(\dfrac{1}{2}\right)^3.1 = \dfrac{1}{8}$

The probability distribution is

X	0	1	2	3
P(x)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

(b) Given that

$$A = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -2 & 5 & 3 \end{bmatrix}$$

$$A' = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 5 \\ 0 & 3 & 3 \end{bmatrix}$$

$$\therefore A + A' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -2 & 5 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 5 \\ 0 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1+1 & 2+0 & 0+(-2) \\ 0+2 & 1+1 & 3+5 \\ -2+0 & 5+3 & 3+3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & 8 \\ -2 & 8 & 6 \end{bmatrix}$$

$$(A+A')' = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & 8 \\ -2 & 8 & 6 \end{bmatrix} = A + A'$$

So $A + A'$ is a symmetric matrix.

$$A - A' = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3 \\ -2 & 5 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & -2 \\ 2 & 1 & 5 \\ 0 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & -2 \\ 2 & 2 & 0 \end{bmatrix}$$

$$(A - A')' = \begin{bmatrix} 0 & -2 & -2 \\ 2 & 0 & 2 \\ 2 & -2 & 0 \end{bmatrix}$$

$$= -\begin{bmatrix} 0 & 2 & 2 \\ -2 & 0 & -2 \\ -2 & 2 & 0 \end{bmatrix}$$

$$= -(A - A')$$

Thus $A - A'$ is a skew symmetric matrix.

(c) $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

$A^2 = A \text{ x } A$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} \text{ x } \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1+2.3+3.4 & 1.2+2(-2)+3.2 & 1.3+2.1+3.1 \\ 3.1+(-2).3+1.4 & 3.2+(-2)(-2)+1.2 & 3.3+(-2).1+1.1 \\ 4.1+2.3+1.4 & 4.2+2.(-2)+1.2 & 4.3+2.1+1.1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+12 & 2-4+6 & 3+2+3 \\ 3-6+4 & 6+4+2 & 9-2+1 \\ 4+6+4 & 8-4+2 & 12+2+1 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix}$$

$A^3 = A^2 \text{ x } A$

$$= \begin{bmatrix} 19 & 4 & 8 \\ 1 & 12 & 8 \\ 14 & 6 & 15 \end{bmatrix} \text{ x } \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 19.1+4.3+8.4 & 19.2+4(-2)+8.2 & 19.3+4.1+8.1 \\ 1.1+12.3+8.4 & 1.2+12(-2)+8.2 & 1.3+12.1+8.1 \\ 14.1+6.3+15.4 & 14.2+6(-2)+15.2 & 14.3+6.1+15.1 \end{bmatrix}$$

$$= \begin{bmatrix} 19+12+32 & 38-8+16 & 57+4+8 \\ 1+36+32 & 2-24+16 & 3+12+8 \\ 14+18+60 & 28-12+30 & 42+6+15 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 13 \\ 92 & 46 & 63 \end{bmatrix}$$

$\therefore A^3 - 23A - 40I$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - 23\begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix} - 40\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 63 & 46 & 69 \\ 69 & -6 & 23 \\ 92 & 46 & 63 \end{bmatrix} - \begin{bmatrix} 23 & 46 & 69 \\ 69 & -46 & 23 \\ 92 & 46 & 23 \end{bmatrix} - \begin{bmatrix} 40 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 40 \end{bmatrix}$$

$$= \begin{bmatrix} 63-23-40 & 46-40-0 & 69-69-0 \\ 69-69-0 & -6+46-40 & 23-23-0 \\ 92-92-0 & 46-46-0 & 63-23-40 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

(d) $\begin{vmatrix} x+1 & w & w^2 \\ w & x+w^2 & 1 \\ w^2 & 1 & x+w \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} x+1+w+w^2 & w & w^2 \\ x+1+w+w^2 & x+w^2 & 1 \\ x+w+w^2+1 & 1 & x+w \end{vmatrix} = 0$$

$(c_1 \to c_1 + c_2 + c_3)$

$$\Rightarrow \begin{vmatrix} x+0 & w & w^2 \\ x+0 & x+w^2 & 1 \\ x+0 & 1 & x+w \end{vmatrix} = 0$$

$$\Rightarrow x\begin{vmatrix} 1 & w & w^2 \\ 1 & x+w^2 & 1 \\ 1 & 1 & x+w \end{vmatrix} = 0$$

$\Rightarrow x.x^2 = 0$

$\Rightarrow x^3 = 0$

$\Rightarrow x = 0$

(e) A person takes 4 tests of succession. Given that the probability of his passing the first test is p. Then the probability of his passing each succeding test is p or $\frac{p}{2}$ depending on his passing or failing the preceding test.

Let S and F denotes the success and failure in the test

P(S) = p, P(F) = 1 - p.

The probability of passing just 3 tests

= P (passing just 3 tests)

= P (SSSF, SSFS, SFSS, FSSS)

= P (SSSF + P (SSFS)+P(SFSS)+P(FSSS)

$= ppp(1-p) + pp(1-p)\frac{p}{2} + p(1-p)\frac{p}{2}p + (1-p)\frac{p}{2}.p.p$

$= p^3(1-p) + \frac{1}{2}p^3(1-p) + \frac{1}{2}p^3(1-p) + \frac{1}{2}p^3(1-p)$

$= \frac{5}{2}p^3(1-p)$

4. (a) The equation of the curve is

$x^2 + y^2 - 4xy + 2 = 0$ ...(1)

Differentiating both sides w.r.t.x, are get

$$\frac{dx^2}{dx} + \frac{dy^2}{dx} - 4d\frac{(xy)}{dx} + \frac{d2}{dx} = 0$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} - 4(x\frac{dy}{dx} + y) + 0 = 0$$

$$\Rightarrow x + y\frac{dy}{dx} - 2(x\frac{dy}{dx} + y) = 0$$

$$\Rightarrow x + y\frac{dy}{dx} - 2x\frac{dy}{dx} - 2y) = 0$$

$$\Rightarrow (y-2x)\frac{dy}{dx} = 2y - x$$

$$\Rightarrow \frac{dy}{dx} = \frac{2y-x}{y-2x}$$

Slope of the normal $= -\frac{1}{\frac{dy}{dx}} = -\frac{y-2x}{2y-x}$

Since the normal is parallel to x-axis, its slope = 0

$$\Rightarrow -\frac{y-2x}{2y-x} = 0$$

$\Rightarrow y - 2x = 0$

$\Rightarrow y = 2x$ ... (2)

Solving (1) and (2), we get

$x^2 + 4x^2 - 8x^2 + 2 = 0$

$\Rightarrow -3x^2 = -2$

$\Rightarrow x^2 = \frac{2}{3}$

$\Rightarrow x = \pm\sqrt{\frac{2}{3}}$

$y = 2x = \pm 2\sqrt{\frac{2}{3}}$

(b) The given function is $y = \frac{\ln x}{x}$ ...(1)

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{x}\right)$$

$$= \frac{x\frac{d\ln x}{dx} - \ln x.\frac{dy}{dx}}{x^2}$$

$$= \frac{x.\frac{1}{x} - \ln x - 1}{x^2} = \frac{1-\ln x}{x^2}$$

If the function is increasing then $\frac{dy}{dx} > 0$

$$\Rightarrow \frac{1-\ln x}{x^2} > 0$$

$\Rightarrow 1 - \ln x > 0$

$\Rightarrow 1 > \ln x$

$\Rightarrow \ln x < 1 = \ln e$

$\Rightarrow x < e$

If the function is decreasing then $\frac{dy}{dx} < 0$

$$\Rightarrow \frac{1-\ln x}{x^2} < 0$$

$\Rightarrow 1 - \ln x < 0$

$\Rightarrow \ln x > 1 = \ln e$

$\Rightarrow x > e.$

(c) $y = e^{x^{e^{x^{e^{x^{\cdots}}}}}}$

$\Rightarrow y = e^{x^y}$

$\therefore \ln y = \ln e^{x^y} = x^y \ln e = x^y$

$\ln \ln y = \ln x^y = y \ln x$

Differentiating both sides w.r t x, we get

$$\frac{d\ln\ln y}{dx} = \frac{d(y\ln x)}{dx}$$

$$\Rightarrow \frac{d\ln(\ln y)}{d\ln y}.\frac{d\ln y}{dy}.\frac{dy}{dx} = y.\frac{d\ln x}{dx} + \ln x\frac{dy}{dx}$$

$$\Rightarrow \frac{1}{\ln y}.\frac{1}{y}.\frac{dy}{dx} = y.\frac{1}{x} + \ln x.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y\ln y} - \ln x\right)\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{1 - y\ln y\ln x}{y\ln y}.\frac{dy}{dx} = \frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2\ln y}{x[1 - y\ln y\ln x]}$$

(d) $x = a\cos\theta$ and $y = b\sin\theta$

$$\therefore \frac{dx}{d\theta} = \frac{d(a\cos\theta)}{d\theta} = a(-\sin\theta) = -a\sin\theta$$

$$\frac{dy}{d\theta} = \frac{d(b\sin\theta)}{d\theta} = b.\cos\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{b\cos\theta}{-a\sin\theta} = -\frac{b}{a}\cos\theta$$

$$\frac{d^2y}{dx^2} = -\frac{b}{a}.\frac{d\cot\theta}{dx}$$

$$= -\frac{b}{a}.\frac{d\cot\theta}{d\theta}.\frac{d\theta}{dx}$$

$$= -\frac{b}{a}.(-\operatorname{cosec}^2\theta).\frac{1}{-a\sin\theta}$$

$$= -\frac{b}{a^2}\operatorname{cosec}^3\theta.$$

(e) Given function is $f(x) = x^3 - 2x^2 - x + 3$

The given interval is [1, 2].

Since the function is a polynomial, it is continuous and derivable for all $x \in R$.

$f'(x) = 3x^2 - 4x - 1$

Hence (i) f(x) is continuous in [1, 2]

(ii) f(x) is differentiable in (1, 2).

The conditions of Lagranges theorem is satisfied. So there exist at least one real number c in (1, 2) such that

$$f'(c) = \frac{f(2) - f(1)}{2-1}$$

$$\Rightarrow 3c^2 - 4c - 1 = \frac{1-1}{1}$$

$\Rightarrow 3c^2 - 4c - 1 = 0$

$\Rightarrow c^2 = \frac{2+\sqrt{7}}{3}, \frac{2-\sqrt{7}}{3}$

The value of $c = \frac{2-\sqrt{7}}{3}$ is rejected.

The value of C is in [1, 2].

So Langrange's mean value theorem is satisfied.

5. (a) Equation curve is

$y = ae^{3x} + be^{5x}$ ...(1)

$y' = 3ae^{3x} + 5be^{5x}$ ...(2)

$y'' = 9ae^{3x} + 25be^{5x}$ ...(3)

From (1) and (2), we get

$\frac{1}{2}(3y - y') = be^{5x}$ ...(4)

Also $\frac{1}{2}(5y - y') = ae^{3x}$ ...(5)

Thus from (3), (4) and (5)

$y'' = 9.\frac{1}{2}(5y - y') + 25.\frac{1}{2}(3y - y')$

$\Rightarrow 2y'' = 9(5y - y) + 25(3y - y')$

$\Rightarrow 2y'' = 45y - 9y' + 75y - 25y'$

$\Rightarrow 2y'' = 120y - 34y'.$

Which is the required differential equation.

(b) Given differential equation is

$(x^2 + 7x + 12)dy + (y^2 - 6y + 5)dx = 0$

$\Rightarrow \frac{dy}{y^2 - 6y + 5} + \frac{dx}{x^2 + 7x + 12} = 0$

Integrating both sides, we get

$\int \frac{dy}{y^2 - 6y - 5} + \int \frac{dx}{x^2 + 7x + 12} = \text{Constant}$

$\Rightarrow \int \frac{dy}{y^2 - 2y.3 + 9 - 4} + \int \frac{dx}{x^2 + 2.x.\frac{7}{2} + \frac{49}{4} - \frac{1}{4}} = C$

$\Rightarrow \int \frac{dy}{(y-3)^2 + 2^2} + \int \frac{dx}{\left(x + \frac{7}{2}\right)^2 - \left(\frac{1}{2}\right)^2} = C$

$\Rightarrow \frac{1}{2.2} \ln\left(\frac{y - 3 - 2}{y - 3 + 2}\right) + \frac{1}{2.\frac{1}{2}} \ln\left(\frac{x + \frac{7}{2} - \frac{1}{2}}{x + \frac{7}{2} + \frac{1}{2}}\right) = C$

$\Rightarrow \frac{1}{4}\ln\left(\frac{y-5}{y-1}\right) + \ln\left(\frac{x+6}{x+4}\right) = C.$

(c) Let $I = \int \frac{2x+1}{\sqrt{x^2 + 10x + 29}} dx$

$= \int \frac{(2x+10) - 9}{\sqrt{x^2 + 10x + 29}} dx$

$= \int \frac{2x+10}{\sqrt{x^2 + 10x + 29}} dx - 9\int \frac{1}{\sqrt{x^2 + 10x + 29}} dx$

$= I_1 - I_2$ (say) ... (1)

For I_1, Let $x^2 + 10x + 29 = t$

$\therefore (2x + 10)dx = dt$

$I_1 = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}} dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C$

$= 2\sqrt{t} + C$

$= 2\sqrt{x^2 + 10x + 29} + C$

$I_2 = 9\int \frac{1}{\sqrt{x^2 + 10x + 29}} dx$

$= 9\int \frac{1}{\sqrt{x^2 + 2.x.5 + 25 + 4}} dx$

$= 9\int \frac{1}{\sqrt{(x+5)^2 + 2^2}} dx$

$= 9\ln[x + 5 + \sqrt{x^2 + 10x + 29}] + C$

From (i) we get

$I = 2\sqrt{x^2 + 10x + 29} + 9\ln[x + 5 + \sqrt{x^3 + 10x + 29}] + C$

(d) Let $I=\int_0^{\pi}\frac{\text{Cos}x\,dx}{(2-\text{Sin}\,x)(3-\text{Sin}\,x)}$

Let Sin x = t

$\Rightarrow$ Cosx dx = dt

When x = 0, t = Sin 0 = 0

When $x=\frac{\pi}{2},\ t=\text{Sin}\frac{\pi}{2}=1$

$$I=\int_0^1\frac{dt}{(2-t)(3+t)}$$

Let $\frac{1}{(2-t)(3+t)}=\frac{A}{2-t}+\frac{B}{3+t}$ ... (1)

$\Rightarrow$ 1 = A (3 + t) + B (2 - t).

when t = 2, I = A.5 + B.0 $\Rightarrow A=\frac{1}{5}$

when t = -3, I = A.0 + B.5 $\Rightarrow B=-\frac{3}{5}$

From (1), we get

$$\frac{1}{(2-t)(3+t)}=\frac{1}{5}.\frac{1}{2-t}-\frac{3}{5}.\frac{1}{3+t}$$

$$I=\int_0^1\frac{dt}{(2-t)(3+t)}=\frac{1}{5}\int_0^1\frac{1}{2-t}dt-\frac{3}{5}\int_0^1\frac{1}{3+t}dt$$

$$=-\frac{1}{5}[\ln 1-\ln 2]-\frac{3}{5}[\ln 4\ln 3]$$

$$=\frac{1}{5}\ln 2-\frac{3}{5}\ln\left(\frac{4}{3}\right).$$

(e) The equation of the curve is

y = 6x - x^2 ... (1)

The shape of curbve is drawn from the following table.

X	0	1	2	3	4	5	6
y	0	5	8	9	8	5	0

Required area

$$=\int_0^6 y\,dx$$

$$=\int_0^6(6x-x^2)dx$$

$$=\left[6.\frac{x^2}{2}-\frac{x^3}{3}\right]_0^6$$

= 108 - 72 - 0

=36 sq. unit.

6. (a) Let OA and OB be two mutually perpendicular lines whose direction cosines are $\langle l_1,m_1,n_1\rangle$ and $\langle l_2,m_2,n_2\rangle$ respectively.

Let θ be the angle between OA & OB. Let OC be a straight line whose direction cosines are $\langle l,m,n\rangle$ and which is perpendicular to OA and OB.

Since OC is perpendicular to OA and OB, we have

$$ll_1+mm_1+nn_1=0$$

$$ll_2+mm_2+nn_2=0$$

$$\therefore\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{m_1n_2-m_2n_1}$$

$$=\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(m_1n_2-m_2n_1)^2}}$$

$$\therefore\frac{1}{\text{Sin}\,90}=1$$

$$\Rightarrow l=m_1n_2-m_2n_1$$

$$m=n_1l_2-n_2l_1$$

$$n=m_1n_2-m_2n_1$$

Thus the direction consine of OC are

$\langle m_1n_2-m_2n_1,\ n_1l_2-n_2l_2, m_1n_2-m_2n_1\rangle$.

(b) The given plane is

$2x + y + z = 2$ ... (1)

Given line is $\frac{x-2}{1} = \frac{y}{-1} = \frac{z-1}{2} = r$ (say)...(2)

Any point on the line is (r + 2, -r, 2r + 1)

Let P be the point of intersection of (1)&(2).

The coordinates of P are also (r+2, -r, 2r+1) for some value of r.

This point will satisfy the plane (1).

$2(r + 2) - r + 2r + 1 = 2$

$\Rightarrow 2r + 4 - r + 2r + 1 = 2$

$\Rightarrow 3r + 5 = 2$

$\Rightarrow 3r = -3$

$\Rightarrow r = -1$

The point of intersection of (1) and (2) is (-1 + 2, - (-1), -2 + 1) = (1, 1, -1)

(c) Given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

and $\vec{b} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\therefore \vec{a} + \vec{b} = (\hat{i} + \hat{j} + \hat{k}) + (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 2\hat{i} + 3\hat{j} + 4\hat{k}$

$\vec{a} - \vec{b} = \hat{i} + \hat{j} + \hat{k} - (\hat{i} + 2\hat{j} + 3\hat{k})$

$= 0\hat{i} - \hat{j} - 2\hat{k}$

A vector perpendicular to $\vec{a} + \vec{b}$ & $\vec{a} - \vec{b}$ is

$(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times (0\hat{i} - \hat{j} - 2\hat{k})$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 3 & 4 \\ 0 & -1 & -2 \end{vmatrix}$$

$= \hat{i}\,(-6 + 4) - \hat{j}\,(-4 - 0) + \hat{k}\,(-2 - 0)$

$= -2\hat{i} + 4\hat{j} - 2\hat{k}$

$|(\vec{a} + \vec{b}) \times (\vec{a} - \vec{b})| = |-2\hat{i} + 4\hat{j} - 2\hat{k}|$

$= \sqrt{(-2)^2 + 4^2 + (-2)^2}$

$= \sqrt{4 + 16 + 4} = \sqrt{24} = 2\sqrt{6}$

Required unit vector $\frac{-2\hat{i} + 4\hat{j} - 2\hat{k}}{2\sqrt{6}}$

$= \frac{1}{\sqrt{6}}(-\hat{i} + 2\hat{j} - \hat{k})$.

(d) Let θ be the angle between $\vec{a}$ and $\vec{b}$,

$\vec{a}.\vec{b} = |\vec{a}||\vec{b}|\cos\theta$

$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin\theta, \hat{n}$ where $\hat{n}$ is an unit vector perpendicular to $\vec{a}$ & $\vec{b}$.

$(\vec{a} \times \vec{b})^2 = |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta.\hat{n}^2$

$= |\vec{a}|^2 |\vec{b}|^2 \sin^2\theta \qquad \hat{n}^2 = \hat{n}.\hat{n} = 1$

$= |\vec{a}|^2 |\vec{b}|^2 (1 - \cos^2\theta)$

$= a^2 b^2 - (|\vec{a}||\vec{b}|\cos\theta)^2$

$= a^2 b^2 - (\vec{a}.\vec{b})^2$.

(e) Given that $\vec{n} = 2\hat{j} + 3\hat{j} - 6\hat{k}$

$|\vec{n}| = \sqrt{2^2 + 3^2 + (-6)^2}$

$= \sqrt{4 + 9 + 36} = \sqrt{49} = 7$

Unit normal to the plane is

$\hat{n} = \frac{\vec{n}}{|\vec{n}|} = \frac{1}{7}(2\hat{i} + 3\hat{j} + 4\hat{k})$

$p = 3.$

Vector equation of the plane is

$\vec{r}.\hat{n} = p$

$\Rightarrow \vec{r}\,.\,\frac{1}{7}(2\hat{i} + 3\hat{j} - 4\hat{k}) = 3$

$\Rightarrow \vec{r}\,.\,(2\hat{i} + 3\hat{j} - 4\hat{k}) = 21$

$\Rightarrow 2x + 3y + 4z = 21.$

GROUP - C

7. (a) $e^{\frac{y}{x}}=\frac{x}{a+bx}$

$\ln e^{\frac{y}{x}}=\ln\left(\frac{x}{a+bx}\right)$

$\Rightarrow \frac{y}{x}\ln e=\ln\left(\frac{x}{a+bx}\right)$ ... (1)

$\Rightarrow \frac{y}{x}=\ln x-\ln(a+bx)$

$\Rightarrow y=x\ln x-x\ln(a+bx)$

Differentialing both sides w.r.t x, we get

$\frac{dy}{dx}=\frac{d(x\ln x)}{dx}-\frac{d[x\ln(a+bx)]}{dx}$

$=x\frac{d\ln x}{dx}+\ln x.\frac{dx}{dx}-\left[x.\frac{d\ln(a+bx)}{dx}+\ln(a+bx)\frac{dx}{dx}\right]$

$=1+\ln x-\frac{bx}{a+bx}-\ln(a+bx)$

$=\left(1-\frac{bx}{a+bx}\right)+\ln x-\ln(a+bx).$

$=\ln\frac{x}{a+bx}+1-\frac{bx}{a+bx}$ ... (2)

$=\frac{y}{x}+1-\frac{bx}{a+bx}$

$\Rightarrow x\frac{dy}{dx}=y+x-\frac{bx^2}{a+bx}$

$x\frac{dy}{dx}-y=x\left(1-\frac{bx}{a+bx}\right)$

$=x\left(\frac{a+bx-bx}{a+bx}\right)$

$=\frac{ax}{a+bx}$

$\Rightarrow\left(x\frac{dy}{dx}-y\right)^2=\frac{a^2x^2}{(a+bx)^2}$ ... (3)

Again from (2), we get

$\frac{dy}{dx}=\ln x-\ln(a+bx)+\frac{a}{a+bx}$

$\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{1}{x}-\frac{b}{a+bx}+a\frac{(-b)}{(a+bx)^2}$

$=\frac{1}{x}-\frac{b}{a+bx}-\frac{ab}{(a+bx)^2}$

$=\frac{a+bx-bx}{x(a+bx)}-\frac{ab}{(a+bx)^2}$

$=\frac{a}{x(a+bx)}-\frac{ab}{(a+bx)^2}$

$=\frac{a}{x(a+bx)}\left[\frac{1}{x}-\frac{b}{a+bx}\right]$

$=\frac{a}{a+bx}.\frac{a+bx-bx}{x(a+bx)}$

$=\frac{a^2}{x(a+bx)^2}$

$\therefore x^3\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{a^2x^2}{(a+bx)^2}$ ... (4)

From (3) and (4), we get

$x^3\frac{d}{dx}\left(\frac{dy}{dx}\right)=\left(x\frac{dy}{dx}-y\right)^2.$

(b) The equation of the curve is

$ax^2=y^3$ ... (1)

The given point is A(0, 8a).

Let P(x, y) be any point on the curve $ax^2=y^3$.

$PA=\sqrt{(x-0)^2+(y-8a)^2}$

$=\sqrt{x^2+(y-8a)^2}$

Let $s=PA^2=x^2+(y-8a)^2$

$=\frac{y^3}{a}+(y-8a)^2$

$\frac{ds}{dy}=\frac{3y^2}{a}+2(y-8a)$

$$\frac{d^2s}{dy^2}=\frac{6y}{a}+2$$

For maximum or minimum, $\frac{ds}{dy}=0$

$$\Rightarrow \frac{3y^2}{a}+2(y-8a)=0$$

$$\Rightarrow \frac{3y^2}{a}+2y-16a=0$$

$$\Rightarrow 3\left(\frac{y}{a}\right)^2+2.\frac{y}{a}-16=0$$

$$\frac{y}{a}=\frac{-2\pm\sqrt{4-4.3(-16)}}{2.3}$$

$$=\frac{-2\pm\sqrt{196}}{6}$$

$$=\frac{12}{6},\frac{16}{6}$$

$$=2,-\frac{8}{3}$$

$$\Rightarrow y=2a,-\frac{8}{3}$$

Since y is non negative, y = 2a.

When $y=2a, \frac{d^2s}{dy^2}=\frac{6.2a}{a}+2=14>0$

$\therefore$ s is minimum when $y=2a$.

The shortest distance of

$$P=\sqrt{\frac{y^3}{a}+(y-8a)^2}$$

$$=\sqrt{\frac{8a^3}{a}+(2a-8a)^2}$$

$$=\sqrt{8a^2+36a^2}$$

$$=\sqrt{44a^2}$$

$$=2a\sqrt{11}\cdot$$

8. (a) The given circle is $x^2+y^2=2x$

$$\Rightarrow x^2-2x+1+y^2=1$$

$$\Rightarrow (x-1)^2+y^2=1 \quad \text{... (1)}$$

The centre of the circle is (1, 0) and radius = 1

Equation of the parabola is

$$y^2=x \quad \text{... (2)}$$

Solving (1) and (2), we get

$$x^2+2=2x$$

$$\Rightarrow x^2-x=0$$

$$\Rightarrow x(x-1)=0$$

$$\Rightarrow x=0,1.$$

Area common to the parabola and ellipse

= 2 x Area of OABCO

= 2 [Area of OAC+Area of ABC]

$$=2\left[\int_0^1\sqrt{xdx}+\int_1^2\sqrt{2x-x^2}.dx\right]$$

$$=2.\left[\frac{x^{3/2}}{3/2}\right]_0^1+2\int_0^1\sqrt{1-t^2}dt \text{ [where } x-1=t]$$

$$=\frac{4}{3}+2\left[\frac{t\sqrt{1-t^2}}{2}+\text{Sin}^{-1}t\right]_0^1$$

$$=\frac{4}{3}+2\left[0+\frac{1}{2}\text{Sin}^{-1}1-0\right].$$

$$=\frac{4}{3}+2.\frac{1}{2}.\frac{\pi}{2}=\frac{4}{3}+\frac{\pi}{2} \text{ sq. unit.}$$

(b) The given equation is

$$y^2+x^2\frac{dy}{dx}=xy\frac{dy}{dx}$$

$$\Rightarrow (xy-x^2)\frac{dy}{dx}=y^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2}{xy-x^2} \quad \text{... (1)}$$

Let y = vx

$$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

From (1), we get

$$v + x\frac{dv}{dx} = \frac{v^2x^2}{x.vx - x^2}$$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2}{v-1}$$

$$\Rightarrow x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1} = \frac{v}{v-1}$$

$$\Rightarrow \frac{v-1}{v}.dv = \frac{dx}{x}$$

$$\Rightarrow \left(1 - \frac{1}{v}\right)dv = \frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(1 - \frac{1}{v}\right)dv = \int\frac{dx}{x}$$

$\Rightarrow$ v - ln v = ln x + C

$$\Rightarrow \frac{y}{x} - \ln\frac{y}{x} = \ln x + C.$$

(c) Let $I = \int\frac{dx}{2\cos^2 x + 3\cos x} = \int\frac{1}{\cos x(2\cos x + 3)}dx$

$$= \int\left(\frac{1}{3}.\frac{1}{\cos x} - \frac{2}{3}.\frac{1}{2\cos x + 3}\right)dx$$

$$= \frac{1}{3}\int \sec x\, dx - \frac{2}{3}\int\frac{1}{2\cos x + 3}dx$$

$$= \frac{1}{3}\ln(\sec x + \tan x) - \frac{2}{3}\int\frac{1}{2\cos x + 3}dx \quad ...(1)$$

$$\int\frac{1}{2\cos x + 3}dx = \int\frac{1}{2\left(\dfrac{1 - \tan^2\frac{x}{2}}{1 + \tan^2\frac{x}{2}}\right) + 3}$$

$$= \int\frac{1 + \tan^2\frac{x}{2}}{2(1 - \tan^2\frac{x}{2}) + 3(1 + \tan^2\frac{x}{2})}$$

$$= \int\frac{\sec^2\frac{x}{2}dx}{\tan^2\frac{x}{2} + 5} = I_1 \quad \text{(say) ...(2)}$$

Let $\tan\frac{x}{2} = t$

$$\frac{1}{2}\sec^2\frac{x}{2}dx = dt$$

$$\Rightarrow \sec^2\frac{x}{2}dx = 2dt$$

$$I_1 = \int\frac{2dt}{t^2 + 5} = 2\int\frac{dt}{t^2 + (\sqrt{5})^2}$$

$$= 2.\frac{1}{\sqrt{5}}\tan^{-1}\frac{t}{\sqrt{5}} + C$$

$$= \frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan\frac{x}{2}\right) + C$$

From (1), we get

$$I = \frac{1}{3}\ln(\sec x + \tan x) - \frac{2}{3}.\frac{2}{\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan\frac{x}{2}\right) + C$$

$$= \frac{1}{3}\ln(\sec x + \tan x) - \frac{4}{3\sqrt{5}}\tan^{-1}\left(\frac{1}{\sqrt{5}}\tan\frac{x}{2}\right) + C.$$

9. (a) Let A, B, C, D be four points whose coordinates are (6, 2, -1), (2, -1, 3), (-1, 2, -4) and (-12, -1, -3) respectively.

First we shall find the equation of the plane which is passing through three points A (6, 2, -1), B (2, -1, 3), C (-1, 2, -4).

This plane is passing through the point A (6, 2, -1) whose position vector is

$\vec{a} = 6\hat{i} + 2\hat{j} - \hat{k}$.

The plane is normal to the vector $\vec{n}$ where $\vec{n} = \overrightarrow{AB} \text{ x } \overrightarrow{AC}$.

$\overrightarrow{AB}$ = P.V. of B - P.V. of A

$$(2\hat{i} - \hat{j} + 3\hat{k}) - (6\hat{i} = 2\hat{j} - \hat{k})$$

$$= -4\hat{i} - 3\hat{j} + 4\hat{k}$$

$$\overrightarrow{AC} = (-\hat{i} + 2\hat{j} - 4\hat{k}) - (6\hat{i} + 2\hat{j} - \hat{k})$$

$$= -7\hat{i} + 0\hat{j} - 3\hat{k}$$

$$\vec{n} = \overrightarrow{AB} \text{ x } \overrightarrow{AC}$$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -4 & -3 & 4 \\ -7 & 0 & -3 \end{vmatrix}$$

$$= \hat{i}(9-0) - \hat{j}(12+28) + \hat{k}(0-21)$$

$$= 9\hat{i} - 40\hat{j} - 21\hat{k}$$

Vector equation of the plane passing through A, B & C is

$$\vec{r}.\vec{n} = \vec{a}.\vec{n}$$

$$\vec{r}.(9\hat{i} - 40\hat{j} - 21\hat{k})$$

$$= (6\hat{i} + 2\hat{j} - \hat{k}) - (9\hat{i} - 40\hat{j} - 20\hat{k})$$

$$\Rightarrow \vec{r}.(9\hat{i} - 40\hat{j} - 21\hat{k}) = 54 - 80 + 21$$

$$\Rightarrow \vec{r}.(9\hat{i} - 40\hat{j} - 21\hat{k}) = -5$$

$$\Rightarrow (x\hat{i} - y\hat{j} + z\hat{k}).(9\hat{i} - 40\hat{j} - 21\hat{k}) = -5 \text{ ...(1)}$$

Let us verify whether the point D (-12,-1,-3) satisfy the equation (1).

$$(-12\hat{i} - \hat{j} - 3\hat{k}).(9\hat{i} - 40\hat{j} - 21\hat{k})$$

$$= -108 + 40 + 63$$

$$= -5$$

So four given points are coplanar.

(b) Equations of two given lines are

$$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7} = k\,(\text{say}) \quad \text{... (1)}$$

$$\frac{x+2}{2} = \frac{y-3}{-3} = \frac{z-4}{8} \quad \text{... (2)}$$

Let P be the given point (1, -1, -10).

Let PM be the distance of P from the line (1) measured parallel to the line (2).

Since PM us parallel to the line (2), its direction ratios are <2, -3, 8>.

Any point on the line (1) is

(k+4, -4k-3, 7k-1).

The coordinates of M are also

(k+4, -4k-3, 7k-1) for some value of k.

The d. rs of PM are

<k+4-1, -4k-3+1, 7k-1+10>

= <k+3, -4k-2, 7k-9>

Given that the d. rs of PM are <2, -3, 8>

$$\therefore \quad \frac{k+3}{2} = \frac{-4k-2}{-3} = \frac{7k+9}{8} \quad \text{... (3)}$$

From first two relations, we get

$$\frac{k+3}{2} = \frac{-4k-2}{-3}$$

$$\Rightarrow -3k - 9 = -8k - 4$$

$$\Rightarrow 5k = 5$$

$$\Rightarrow k = 1.$$

The coordinates of M are

$$(1+4, -4-3, 7-1) = (5, -7, 6)$$

$$PM = \sqrt{(5-1)^2 + (-7+1)^2 + (6+10)^2}$$

$$= \sqrt{4^2 + (-b)^2 + 16^2}$$

$$= \sqrt{16 + 36 + 256}$$

$$= \sqrt{308}.$$

10. (a) Let $\sin^{-1}\frac{x}{a} = \alpha$, $\sin^{-1}\frac{y}{b} = \beta$

$$\Rightarrow \sin\alpha = \frac{x}{a}, \sin\beta = \frac{y}{b}$$

$$\therefore \cos\alpha = \sqrt{1-\sin^2\alpha} = \sqrt{1-\frac{x^2}{a^2}}$$

$$\cos\beta = \sqrt{1-\sin^2\beta} = \sqrt{1-\frac{y^2}{b^2}}.$$

Given that $\sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{b} = \sin^{-1}\frac{c^2}{ab}$

$$\Rightarrow \alpha + \beta = \sin^{-1}\frac{c^2}{ab}$$

$$\Rightarrow \cos(\alpha+\beta) = \cos\left(\sin^{-1}\frac{c^2}{ab}\right)$$

$$\Rightarrow \cos\alpha\ \cos\beta - \sin\alpha.\sin\beta$$

$$= \cos\cos^{-1}\sqrt{1-\frac{c^4}{a^2b^2}}$$

$$\Rightarrow \sqrt{1-\frac{x^2}{a^2}}\sqrt{1-\frac{y^2}{b^2}} - \frac{x}{a}.\frac{y}{b} = \sqrt{1-\frac{c^4}{a^2b^2}}$$

$$\Rightarrow \frac{1}{ab}\sqrt{(a^2-x^2)(b^2-y^2)}-\frac{xy}{ab}=\sqrt{\frac{a^2b^2-c^4}{a^2b^2}}$$

$$\Rightarrow \sqrt{(a^2-x^2)(b^2-y^2)}-xy=\sqrt{a^2b^2-c^4}$$

$$\Rightarrow \sqrt{(a^2-x^2)(b^2-y^2)}=xy+\sqrt{a^2b^2-c^4}$$

Squaring both sides, we get

$$(a^2-x^2)(b^2-y^2)=[xy+\sqrt{a^2b^2-c^4}]^2$$

$$\Rightarrow a^2b^2-b^2x^2-a^2y^2+x^2y^2$$

$$=x^2y^2+2xy\sqrt{a^2b^2-c^4}+a^2b^2-c^4$$

$$\Rightarrow b^2x^2+a^2y^2+2xy\sqrt{a^2b^2-c^4}=c^4$$

(b) The given L.P.P. is

Maximize $Z=10x_1+12x_2+8x_3$

Subject to $x_1+2x_2=30$

$5x_1-7x_3\geq 12$

$x_1+x_2+x_3=20$

$x_1,x_2,x_3\geq 0$

Eliminating x_3 from all expression the L.P.P. becomes

Maximize $Z=2x_1+4x_2+160$

Subject to $x_1+2x_2\leq 30$

$12x_1+7x_2\geq 152$

$x_1,\ x_2\geq 0$.

Treating the constraints as equations, we get

$x_1+2x_2=30$ ... (1)

$12x_1+7x_2=152$ ... (2)

To draw the line (1), we have the following table.

x_1	30	0
X_2	0	15

The line (1) pases through the points (30, 0) and (0, 15).

To draw the line (2), we have the following table.

x_1	8	1
X_2	8	20

The line (2) pases through the points (8, 8) and (1, 20).

The shaded portion is the feasible region.

The value of the objective function at different corner points are given in the following table.

Point	x_1	x_2	$Z=2x_1+4x_2+160$
A	$\frac{38}{3}$	0	$Z=2x\frac{38}{3}+4x0+160=\frac{556}{3}$
B	30	0	$Z=Z=2x30+4.0+160=220$
C	$\frac{94}{17}$	$\frac{208}{17}$	$Z=\frac{3740}{17}$

Z is maximum when $x_1=30,\ x_2=0,$

Max. value of Z is 220.

(c) Given that $F:X\to Y$ is injective.

Let A and B are subsets of X.

Let $f(x)\in f(A\cap B)$

$\Leftrightarrow x\in A\cap B$

$\Leftrightarrow x\in A\wedge x\in B$

$\Leftrightarrow f(x)\in f(A)\wedge f(x)\in f(B)$ ($\because$ f is injective)

$\Leftrightarrow f(x)\in f(A)\cap f(B)$

So $f(A\cap B)=f(A)\cap f(B)$

Conversely let $f(A\cap B)=f(A)\cap f(B)$

Let f be not injective

Thus if $f(x)\in f(A\cap B)\Leftrightarrow x\in A\cap B$

$\Leftrightarrow x\in A\wedge x\in B$

$\Leftrightarrow f(x)\in f(A)\wedge f(x)\in f(B)$

$\Leftrightarrow f(x)\in f(A)\cap f(B)$

$\Leftrightarrow f(A\cap B)=f(A)\cap f(B)$ is false.

So f must be injective.

11. (a) The given equations are

x - 2y = 3

3x + 4y - z = - 2

5x - 3z = - 1

$$\Rightarrow \quad x - 2y + 0z = 3$$
$$3x + 4y - z = -2$$
$$5x + 0y - 3z = -1$$

Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 4 & -1 \\ 5 & 0 & -3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, D = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$

The given equations become

$$\begin{bmatrix} 1 & -2 & 0 \\ 3 & 4 & -1 \\ 5 & 0 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow A\ X = D$$

$$\Rightarrow X = A^{-1} D = \frac{\text{adj} A}{|A|} . D \qquad \text{... (1)}$$

$$|A| = \begin{vmatrix} 1 & -2 & 0 \\ 3 & 4 & -1 \\ 5 & 0 & -3 \end{vmatrix}$$

$$= 1 \begin{vmatrix} 4 & -1 \\ 0 & -3 \end{vmatrix} - (-2) \begin{vmatrix} 3 & -1 \\ 5 & -3 \end{vmatrix} + 0$$

= 1 (- 12 - 0) + 2(- 9 + 5)

= - 12 - 8 = - 20 $\neq$ 0

The system has unique solution and consistent.

We shall find adj A.

$$C_{11} = \begin{vmatrix} 4 & -1 \\ 0 & -3 \end{vmatrix} = -12$$

$$C_{12} = -\begin{vmatrix} 3 & -1 \\ 5 & -3 \end{vmatrix} = -(-9+5) = 4$$

$$C_{13} = \begin{vmatrix} 3 & 4 \\ 5 & 0 \end{vmatrix} = 0 - 20 = -20$$

$$C_{21} = -\begin{vmatrix} -2 & 0 \\ 0 & -3 \end{vmatrix} = -(6-0) = -6$$

$$C_{22} = \begin{vmatrix} 1 & 0 \\ 5 & -3 \end{vmatrix} = -3 - 0 = -3$$

$$C_{23} = \begin{vmatrix} 1 & -2 \\ 5 & 0 \end{vmatrix} = -(0+10) = -10$$

$$C_{31} = \begin{vmatrix} -2 & 0 \\ 4 & -1 \end{vmatrix} = 2 - 0 = 2$$

$$C_{32} = -\begin{vmatrix} 1 & 0 \\ 3 & -1 \end{vmatrix} = -(-1-0) = 1$$

$$C_{33} = -\begin{vmatrix} 1 & -2 \\ 3 & 4 \end{vmatrix} = 4 + 6 = 10$$

$$\text{Matrox of cofactors} = \begin{bmatrix} -12 & 4 & -20 \\ -6 & -3 & -10 \\ 2 & 1 & 10 \end{bmatrix}$$

$$\therefore \text{ Adj } A = \begin{bmatrix} -12 & -6 & 2 \\ 4 & -3 & 1 \\ -20 & -10 & 10 \end{bmatrix}$$

From (1), we get

$$x = \frac{1}{-20} \begin{bmatrix} -12 & -6 & 2 \\ 4 & -3 & 1 \\ -20 & -10 & 10 \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ -1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{20} \begin{bmatrix} -36+12-2 \\ 12+6-1 \\ -60+20-10 \end{bmatrix}$$

$$= -\frac{1}{20} \begin{bmatrix} -26 \\ 17 \\ -50 \end{bmatrix} = \begin{bmatrix} \\ - & \frac{26}{20} \\ & \frac{17}{20} \\ & \frac{50}{20} \end{bmatrix}$$

$$\Rightarrow \quad x = \frac{26}{20} = \frac{13}{10}$$

$$y = \frac{17}{20}$$

$$z = \frac{5}{2}$$

(b) Let E_1, E_2, E_3 be the events that a person is a farmer, a person is a businessman, and a person is a service holder respectively. Let A be the event that a person is above the poverty line.

$\therefore \quad P(E_1) = \frac{50}{100} = \frac{5}{10}$

$P(E_2) = \frac{30}{100} = \frac{3}{10}$

$P(E_3) = \frac{20}{100} = \frac{2}{10}$

$P(A/E_1) = \frac{10}{100} = \frac{1}{10}$

$P(A/E_2) = \frac{20}{100} = \frac{2}{10}$

$P(A/E_3) = \frac{50}{100} = \frac{5}{10}$

By the theorem of total probability P (The person is above the poverty line)

= P (A)

$= P(E_1) . P(A/E_1) + P(E_2) P(A/E_2) + P(E_3) P(A/E_3)$

$= \frac{5}{10}.\frac{1}{10} + \frac{3}{10}.\frac{2}{10} + \frac{2}{10}.\frac{5}{10}$

$= \frac{5}{100} + \frac{6}{100} + \frac{10}{100} = \frac{21}{100} = 21\%$

Thus the probability that a villager chosen from the adult population of the village, selected at random above the poverty line is 21%.

(c) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$

We know A = IA

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -2 \\ 0 & -2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$[R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1]$

$$\Rightarrow \begin{bmatrix} 3 & 6 & 9 \\ 0 & -3 & -2 \\ 0 & -6 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 3 \end{bmatrix} A$$

$(R_1 \to 3R_1,\ R_3 \to 3R_3)$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 5 \\ 0 & -3 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 3 \end{bmatrix} A$$

$(R_1 \to R_1 + 2R_3,\ R_3 \to R_3 - 2R_2)$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 12 & -15 \\ 0 & -3 & 6 \\ 1 & -2 & 3 \end{bmatrix} A$$

$(R_1 \to R_1 - 5R_3,\ R_2 \to R_2 + 2R_3)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} A$$

$(R_1 \to \frac{1}{3}R_1,\ R_2 \to \frac{1}{3}R_2)$

Thus $A^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

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2018 (A)

Full Marks : 100 Time : 3 hours

The figures in the right-hand margin indicate marks.
Answer the questions of all the Groups as directed.
Electronic gadgets are not allowed in the Examination Hall

GROUP - A

(Marks : 10)

1. Answer all questions : [1x10=10]

(a) Sets A and B have respectively m and n elements. The total number of relations from A to B is 64. If $m < n$ and $m \neq 1$, write the values of m and n respectively.

(b) Write the principal value of

$$\sin^{-1}\left(-\frac{1}{2}\right)+\cos^{-1}\cos\left(-\frac{\pi}{2}\right)$$

(c) If every element of a third order determinant of value 8 is multiplied by 2, then write the value of the new determinant.

(d) In a Davis Cup tie between India and South Korea, write the probability that India is ahead 2-1 after 3 matches assuming that both the teams are equally likely to win each match.

(e) Write the interval in which the function $f(x)=\sin^{-1}(2-x)$ is differentiable.

(f) A balloon is pumped at the rate of 2 cm^3/ minute. Write the rate of increase of the surface area, when the radius is 0.5 cm.

(g) Write the definite integral which is equal to $\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{\sqrt{n^2+r^2}}$

(h) If p and q are respectively degree and order of the differential equation $y=e^{dy/dx}$, then write the relation between p and q.

(i) If $\left(\vec{a}\times\vec{b}\right)^2+\left(\vec{a}.\vec{b}\right)^2$ write the value of ab.

(j) Write the equations of the line 2x+z-4 = 0 = 2y + z in the symmetrical form.

GROUP - B

(Marks : 60)

2. Answer any three questions : [4x3=12]

(a) Let ~ be defined by (m, n) - (p, q) if mq=np, where n, n, p, q $\in$ Z - {0}. Show that it is an equivalence relation.

(b) Let $f(x)=\sqrt{x}, g(x)=1-x^2$. Computer f o g and g o f, and find their natural domains.

(c) Show that

$$\sin^{-1}\frac{4}{5}+2\tan^{-1}\frac{1}{3}=\frac{\pi}{2}$$

(d) Show that

$$\sin^{-1}\sqrt{\frac{x-q}{p-q}}=\cos^{-1}\sqrt{\frac{p-x}{p-q}}=\cot^{-1}\sqrt{\frac{p-x}{x-q}}$$

(e) Solve the following LPP graphically :

Minimize Z = 4x + 3y

Subject to 2x + 5y $\geq$ 10

x, y $\geq$ 0

3. Answer any three questions : [4x3=12]

(a) If A, B, C are matrices of order 2x2 each and

$$2A+B+C=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}.\ A+B+C=\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}$$

and $A+B-C=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}$

then find A, B and C.

(b) Find the inverse of the following matrix :

$$\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}$$

(c) Show that

$$\begin{bmatrix}1-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{bmatrix}=(a+b+c)^3$$

(d) A bag A contains 2 white and 3 red balls and another bag B contains 4 white and 5 red balls. One ball is drawn at random from a bag chosen at random and it is found to be red. Find the probability that it was drawn from bag B.

(e) If P(A)=0.6, P(B/A)=0.5, find $P(A \cup B)$ when A and B are independent.

4. Answer any three questions : [4x3=12]

(a) Differentiate

$$y = \tan^{-1}\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}$$

(b) Differentiate

$y = (\sin y)^{2in2x}$

(c) Test the differentiability and continuity of the following function at x = 0 :

$$f(x) = \begin{cases} \frac{1-e^{-x}}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

(d) Show that the sum of intercepts on the coordinate axes of any tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ is constant.

(e) Show that

2 sin x + 3 tan x > 3x for all $x \in \left(0, \frac{\pi}{2}\right)$

5. Answer any three questions : [4x3=12]

(a) Evaluate $\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

(b) Show that : $\int_0^1 \frac{in\,x}{\sqrt{1-x^2}}dx = -\frac{\pi}{2}in2$

(c) Find the are of the region enclosed by the two parabolas $y^2 = 4ax$ and $x^2 = 4ay$.

(d) Form the differential equation whose general solution is $y = a\sin t + be^t$.

(e) Solve the following differential equation:

$(1+y^2)dx + (x - e^{-\tan^{-1}y})dy = 0$

6. Answer any three questions : [4x3=12]

(a) Find the area of the triangle ABC with vertices A (1, 2, 4) B (3, 1, -2) and C (4, 3, 1) by vector method.

(b) Prove that

$$\left[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

(c) If the sum of two unit vecotrs is a unit venctor, show that the magnitude of their difference is $\sqrt{3}$.

(d) Prove that the measure of the angle between two main diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

(e) Two position vectors of two points A and B are $3\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}-2\hat{j}-4\hat{k}$ respectively. Find the equation of the plane passing through B and perpendicular to $\overline{AB}$.

GROUP - C

(Marks : 30)

7. Answer any one question : [6]

(a) If $f: X \to Y$ and $g: Y \to Z$ are two functions, show that g of is invertible if each of f and g is so and then $(g \circ f)^{-1} = f^{-1}og^{-1}$.

(b) If ABC is a right angled triangle at A, prove that

$$\tan^{-1}\frac{b}{a+c} + \tan^{-1}\frac{c}{a+b} = \frac{\pi}{4}$$

where a, b, c are sides of the triangle.

(c) Solve the following LPP graphically :

Maximize

$Z = 3x_1 + 2x_2$

subject to

$-2x_1 + x_2 \leq 1$

$x_1 \leq 2$

$x_1 + x_2 \leq 3$

$x_1, x_2 \geq 0$.

8. Answer any one question : [6]

(a) Solve the following linear algebraic equations using inverse of a matrix :

$$x + y + z = 4$$
$$2x - y + 3z = 1$$
$$3x + 2y - z = 1$$

(b) Two cards are drawn successively without replacement from a well-shuffled deck of 52 cards. Find the probability distribution of the number of aces. Also determine the mean and the variance of the number of aces.

(c) By elementary operations, find A^{-1} for the following :

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$$

9. Answer any one question : [6]

(a) If $x = \dfrac{1-\cos^2\theta}{\cos\theta}, y = \dfrac{1-\cos^{2n}\theta}{\cos^n\theta}$

then show that $\left(\dfrac{dy}{dx}\right)^2 = n^2\left(\dfrac{y^2+4}{x^2+4}\right)$

(b) Find the coordinates of the point on the curve $x^2y - x + y = 0$, where the slope of the tangent is maximum.

10. Answer any one question : [6]

(a) Evaluate $\int \dfrac{2\cos x + 7}{4 - \sin x} dx$

(b) Solve (4x+6y+5) dx - (2x+3y+4) dy = 0

(c) Find the area enclosed by y = 4x - 1 and $y^2 = 2x$.

11. Answer any one question : [6]

(a) If $\vec{a} = 2\hat{i} + \hat{k}, \vec{b} = \hat{i} + \hat{j} + \hat{k}$ and

$\vec{c} = 4\hat{i} + 3\hat{j} + 7\hat{k}$

then find the vector $\vec{r}$ which satisfies

$\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a}$.

(b) Find the shortest distance between the lines

$$\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1} \text{ and}$$

$$\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

Find also the equations of the line of shortest distance.

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ANSWERS

2018 (A) - ANSWERS

GROUP - A

1. (a) Here $|A| = m, |B| = n$

The number of relations from A to B is 2^{mn}.

Given that $2^{mn} = 64 = 2^6$

$\Rightarrow$ mn = 6

Since m $\pm$, $m < n$, so m = 2, n = 3

(b) $\text{Sin}^{-1}\left(-\frac{1}{2}\right) + \text{Cos}^{-1}\text{Cos}\left(-\frac{\pi}{2}\right)$

$= -\frac{\pi}{6} + \text{Cos}^{-1}\text{Cos}\frac{\pi}{2}$

$= -\frac{\pi}{6} + \text{Cos}^{-1}\text{O} = -\frac{\pi}{6} + 1$

Thus the principal value of

$\text{Sin}^{-1}\left(-\frac{1}{2}\right) + \text{Cos}^{-1}\left(-\frac{\pi}{2}\right)$ is $-\frac{\pi}{6} + 1$.

(c) If every element of a third order determinant of value 8 is multiplied by 2 then the value of the new determinant is 64.

(d) The probability that India is ahead 2-1 after three matches $= \frac{2}{3}$.

(e) $\text{Sin}^{-1}(2-x)$ is differentiable

When $-1 \le 2 - x \le 1$

$\Rightarrow -3 \le -x \le -1$

$\Rightarrow 3 \ge x \ge 1$

$\Rightarrow 1 \le x \le 3$.

(f) Let S be the surface area and V be the volume of the ballon.

$\therefore V = \frac{4}{3}\pi r^3$, $S = 4\pi r^2$ where r is the radius of the spherical ballon.

The role of increase of the volume = 2cm³/minute.

$\Rightarrow \frac{dv}{dt} = 2$

$\Rightarrow \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = 2$

$\Rightarrow \frac{4}{3}\pi . 3r^2 \frac{dr}{dt} = 2$

$\Rightarrow 4\pi r^2 \frac{dr}{dt} = 2$ ... (1)

When 4=0.5 cm $= \frac{1}{2}$ cm then from (1), we get

$4 . \pi . \left(\frac{1}{2}\right)^2 \frac{dr}{dt} = 2$

$\Rightarrow \frac{dr}{dt} = \frac{2}{\pi}$

The rate of increase of the surface area is

$\frac{ds}{dt} = \frac{4(4\pi r^2)}{dt} = 4\pi . \frac{dr^2}{dt}$

$= 4\pi . \frac{dr^2}{dr} . \frac{dr}{dt}$

$= 4\pi . 2r . \frac{2}{\pi}$

$= 4\pi . 2 . \frac{1}{2} . \frac{2}{\pi} = 8$ cm²/minute.

(g) $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{r}{\sqrt{n^2 + r^2}}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{r}{n\sqrt{1 + \left(\frac{r}{n}\right)^2}}$

$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{\left(\frac{r}{n}\right)}{n\sqrt{1 + \left(\frac{r}{n}\right)^2}}$ ...(1)

The relation (1) becomes $\int_0^1 \frac{xdx}{\sqrt{1 + x^2}}$.

(h) The given differential equation is

$y = e^{\frac{dy}{dx}}$

$\Rightarrow \text{Log } y = \text{Log} e^{\frac{dy}{dx}} = \frac{dy}{dx} . \log e = \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \log y.$

The degree of the differential equation is 1 = p(sy) and the order of the differential equations 1 = q. Thus p = q = 1.

(i) Given that

$(\vec{a} \times \vec{b})^2 + (\vec{a}.\vec{b})^2 = 144$

$\Rightarrow (ab \sin\theta \,.\, \hat{n})^2 + (ab\cos\theta)^2 = 144$

Where is the angle between $\vec{a}$ & $\vec{b}$

$\Rightarrow a^2b^2 \sin^2\theta + a^2b^2\cos^2\theta = 144$

$\Rightarrow a^2b^2 (\sin^2\theta + \cos^2\theta = 144$

$\Rightarrow (ab)^2 = 144 = 12^2$

$\Rightarrow ab = 12$

(j) The given equations of the line are

$2x + 2 - 4 = 0 = 2y + z$

$2x - 4 = -2$ and $z = -2y$

$\Rightarrow 2(x-2) = -2$ and $z = -2y$

$\Rightarrow x - 2 = \frac{z}{-2} = y$

$\Rightarrow \frac{x-2}{1} = \frac{y-0}{1} = \frac{z-0}{-2}$

Which is required symmetrical form of the line.

GROUP - B

2\. (a) Let $A = Z - \{0\}$

= The set of all non-zero integers.

Let $X = A \times A = \{x, y\} : x, y \in A\}$

Here '~' is a relation such that

for (m, n), (p, q) $\in$ X such that

$(m,n) \sim (p,q) \Leftrightarrow mq = np$

We shall show that '~' is an equivalence relation.

Reflexive

For all $(m,n) \in X$

$(m,n) \sim (m,n)$ as mn = nm

$\Rightarrow$ The relation ~ is an equivalence relation.

Symmetric

For $(m,n), (p,q) \in X$

$(m,n) \sim (p,q) \quad \Rightarrow mq = np$

$\Rightarrow np = mq$

$\Rightarrow pn = qm$

$\Rightarrow (p,q) \sim (m,n)$

$\therefore (m,n) \sim (p,q) \Rightarrow (p,q) \sim (m,n)$

So the relation R is symmetric.

Transitive

For $(m,n), (p,q), (r,s) \in X$

$(m,n) \sim (p,q) \Rightarrow mq = np$ ... (1)

$(p,q) \sim (r,s) \Rightarrow ps = qr$ ... (2)

Multiplying (1) and (2), we get

$mq.ps = np.qr$

$\Rightarrow ms = nr$

$\Rightarrow (m,n) \sim (r,s)$

So the relation '~' is transitive.

Since the relation '~' is reflexive,

Symmetric and transitive, it is an equivalence relation.

(b) Given $f(x) = \sqrt{x}, \quad g(x) = 1 - x^2$

$\therefore (fog)(x) = f[g(x)] = f(1-x^2) = \sqrt{1-x^2}$

$(gof)(x) = g[f(x)] = g(\sqrt{x}) = 1 - x$

Domain of fog is $\{x : -1 \le x \le 1\}$

Domain of gof is R.

(c) $\text{Sin}^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3}$

$= \text{Sin}^{-1}\frac{4}{5} + \text{Sin}^{-1}\frac{2.\frac{1}{3}}{1+\frac{1}{9}}$

$$= \sin^{-1}\frac{4}{5} + \sin^{-1}\left(\frac{\frac{2}{3}}{\frac{10}{9}}\right)$$

$$= \sin^{-1}\frac{4}{5} + \sin^{-1}\left(\frac{3}{5}\right)$$

$$= \sin^{-1}\frac{4}{5} + \cos^{-1}\sqrt{1-\frac{9}{25}}$$

$$= \sin^{-1}\frac{4}{5} + \cos^{-1}\frac{4}{5} = \frac{5}{2}$$

(d) Left term $= \sin^{-1}\sqrt{\frac{x-q}{p-q}}$

$$= \cos^{-1}\sqrt{1-\frac{x-q}{p-q}}$$

$$= \cos^{-1}\sqrt{\frac{p-q-x+q}{p-q}}$$

$$= \cos^{-1}\sqrt{\frac{p-x}{p-q}}$$

Also $\sin^{-1}\sqrt{\frac{x-q}{p-q}} = \cot^{-1}\left(\frac{\sqrt{1-\frac{x-q}{p-q}}}{\sqrt{\frac{x-q}{p-q}}}\right)$

$$= \cot^{-1}\frac{\sqrt{\frac{p-q-x+q}{p-q}}}{\sqrt{\frac{x-q}{p-q}}}$$

$$\cot^{-1}\frac{\sqrt{\frac{p-x}{p-q}}}{\sqrt{\frac{x-q}{p-q}}} = \cot^{-1}\sqrt{\frac{p-x}{x-q}}$$

(e) Given L.P.P is

Mimimize $Z = 4x + 3y$ ... (1)

Subject to $2x + 5y \geq 10$ ... (2)

and $x, y \geq 0$ ... (3)

Converting the inequations to equations, we get

$2x + 5y = 10$ ... (4)

$x = 0,\ y = 0$

When x = 0, y= 2

When y = 0, x = 5

This is shown in the table below

x	0	5
y	2	0

The line (4) interest the axes at (5, 0) and (0, 2).

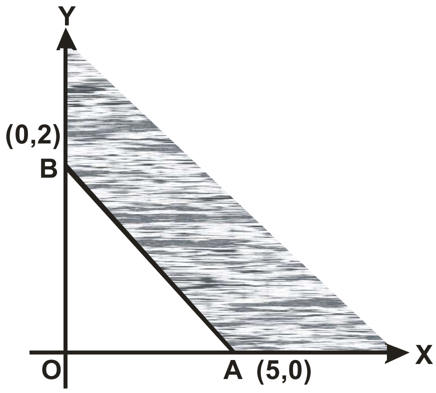
Putting (0, 0) in (2), we get

$2.0 + 5.0 \geq 10$

$\Rightarrow 0 \geq 10$

$\Rightarrow$ The half plane is away from the origin.

The feasible region is as shown in the figure.



The value of Z at the corner points are given in the following table.

Point	x	y	Z = 4x + 3y
A	5	0	Z = 20
B	0	2	Z = 6

Thus the minimum value of Z is 6.

3. (a) A, B and C are matrices of order 2 x 2.

Given that

$$2A + B + C = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \quad \text{... (1)}$$

$$A + B + C = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \quad \text{... (2)}$$

$$A + B + C = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} \quad \text{... (3)}$$

Subtracting (2) from (1), we get

(2A + B + C) - (A + B + C)

$$=\begin{bmatrix}1 & 2\\3 & 0\end{bmatrix}-\begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}$$

$$\Rightarrow A=\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix} \quad \text{... (4)}$$

From (2), we get

$$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}+B+C=\begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}$$

$$\Rightarrow B+C=\begin{bmatrix}0 & 1\\2 & 1\end{bmatrix}-\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}-1 & 0\\2 & 1\end{bmatrix} \quad \text{... (5)}$$

From (3), we get

$$\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}+B-C=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}$$

$$\Rightarrow B-C=\begin{bmatrix}1 & 2\\1 & 0\end{bmatrix}-\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$$

$$=\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix} \quad \text{... (6)}$$

Adding (5) and (6), we get

$$(B+C)+(B-C)=\begin{bmatrix}-1 & 0\\1 & 1\end{bmatrix}+\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$$

$$\Rightarrow 2B=\begin{bmatrix}-1 & 1\\1 & 3\end{bmatrix}$$

$$\Rightarrow B=\frac{1}{2}\begin{bmatrix}-1 & 1\\1 & 3\end{bmatrix}$$

Substracting (6) from (5), we get

$$(B+C)-(B-C)=\begin{bmatrix}-1 & 0\\1 & 2\end{bmatrix}-\begin{bmatrix}0 & 1\\0 & 1\end{bmatrix}$$

$$\Rightarrow 2C=\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}$$

$$\Rightarrow C=\frac{1}{2}\begin{bmatrix}-1 & -1\\1 & 1\end{bmatrix}$$

(b) Let $A=\begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\\1 & 2 & 1\end{bmatrix}$

$$|A|=\begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\\1 & 2 & 1\end{bmatrix}$$

$$=1\,(1-4)-1\,(0-2)+2\,(0=1)$$

$$=-3+2-2=-3$$

C_{11} = Cofactor of

$$1=(-1)^{1+2}\begin{vmatrix}1 & 2\\2 & 1\end{vmatrix}=1-4=-3$$

C_{12} = Cofactor of

$$1=(-1)^{1+2}\begin{vmatrix}0 & 2\\1 & 1\end{vmatrix}=-(0-2)=2$$

$$C_{13}=(-1)^{1+3}\begin{vmatrix}0 & 1\\1 & 2\end{vmatrix}=0-1=-1$$

$$C_{21}=(-1)^{2+1}\begin{vmatrix}1 & 2\\2 & 1\end{vmatrix}=-(1-4)=3$$

$$C_{22}=(-1)^{2+2}\begin{vmatrix}1 & 2\\1 & 1\end{vmatrix}=1-2=-1$$

$$C_{23}=(-1)^{2+3}\begin{vmatrix}1 & 1\\1 & 2\end{vmatrix}=-(2-1)=-1$$

$$C_{31}=(-1)^{3+1}\begin{vmatrix}1 & 2\\1 & 2\end{vmatrix}=0$$

$$C_{32}=(-1)^{3+2}\begin{vmatrix}1 & 2\\0 & 2\end{vmatrix}=-(2-0)=-2$$

$$C_{33}=(-1)^{3+3}\begin{vmatrix}1 & 1\\0 & 1\end{vmatrix}=1$$

Matrix of cofactors $=\begin{bmatrix}-3 & 2 & -1\\3 & -1 & -1\\0 & -2 & 1\end{bmatrix}$

Adjoont of $A = \begin{bmatrix} -3 & 3 & 0 \\ 2 & -1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \text{ adj } (A) = \frac{1}{-3}\begin{bmatrix} -3 & 3 & 0 \\ 2 & -1 & -2 \\ -3-1 & -1 & 1 \end{bmatrix}$$

(c) $\Delta = \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & -a-b-c & 0 \\ 2c & 0 & -a-b-c \end{vmatrix}$$

$(C_2 \to C_2 - C_1$
$C_3 \to C_3 - C_1)$

$$= (a+b+c)\left[(a+b+c)^2 - 0\right]$$

$$= (a+b+c)^3 .$$

(d) The bag 'A' contains 2 white and 3 red balls. The bag B contains 4 white and 5 red balls.

Let E_1 be the event that the ball is drawn from bag A and E_2 be the event that a ball is drawn from the bag B.

Let E be the event that the ball is red,

We shall find $P(E_2 / E)$

Since the probability of choosing each bag is same, we have

$$P(E_1) = P(E_2) = \frac{1}{2}$$

$$P(E/E_1) = \frac{3}{5}, \; P(E/E_2) = \frac{5}{9}$$

According to Bay's theorem,

$$P(E_2/E) = \frac{P(E_2)P(E/E_2)}{P(E_1)P(E/E_1) + P(E_2).P(E/E_2)}$$

$$= \frac{\frac{1}{2}.\frac{5}{9}}{\frac{1}{2}.\frac{3}{5}.\frac{1}{2}.\frac{5}{9}} = \frac{25}{52}.$$

(e) Given that $P(A) = 0.6, P(B/A) = 0.5$

Given that A and B are independent events.

$P(A \cap B) = P(A).P(B)$ ... (1)

We know $P(B/A) = \dfrac{P(B \cap A)}{P(A)}$

$$\Rightarrow 0.5 = \frac{P(A \cap B)}{0.6}$$

$$\Rightarrow P(A \cap B) = 0.5 \text{ x } 0.6 = 0.3$$

From (1), we get

$$0.3 = 0.6 \text{ x} P(B)$$

$$P(B) = \frac{0.3}{0.6} = \frac{3}{6} = \frac{1}{2} = 0.5$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$= 0.6 + 0.5 - 0.3$$

$$= 1.1 - 0.3 = 0.8$$

4. (a) Let $y = \tan^{-1}\left[\dfrac{\sqrt{1+x^2} + \sqrt{1-x^2}}{\sqrt{1+x^2} - \sqrt{a-x^2}}\right]$

Let $x^2 = \text{Cos}\,\theta \Rightarrow \theta = \cos^{-1} x^2$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta} + \sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta - \sqrt{1-\cos\theta}}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\frac{\theta}{2}} + \sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}} - \sqrt{2\sin^2\frac{\theta}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{\sqrt{2}\cos\frac{\theta}{2} + \sqrt{2}\sin\frac{\theta}{2}}{\sqrt{2}\cos\frac{\theta}{2} - \sqrt{2}\sin\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\left(\frac{\tan\frac{\pi}{4} + \tan\frac{\theta}{2}}{1-\tan\frac{\pi}{4}.\tan\frac{\theta}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right) = \frac{\pi}{4}+\frac{1}{2}\theta$$

$$= \frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2\right)$$

$$= \frac{d(\pi/4)}{dx} + \frac{1}{2}\frac{d\cos^{-1}x^2}{dx}$$

$$= 0 + \frac{1}{2}\frac{d\cos^{-1}x^2}{dx^2}.\frac{dx^2}{dx}$$

$$= \frac{1}{2}x - \frac{1}{\sqrt{1-x^4}}x\,2x$$

$$= -\frac{x}{\sqrt{1-x^4}}$$

(b) $y = (\sin y)^{\sin 2x}$

$$\Rightarrow \log y = \log(\sin y)^{\sin 2x}$$

$$= \sin 2x \log(\sin y)$$

Differentiating both sides w.r.t x, we get

$$\frac{d\log y}{dx} = \frac{d\sin 2x\log(\sin y)}{dx}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \sin 2x\frac{d\log\sin y}{dx}$$

$$+\log\sin y\frac{d\sin 2x}{dx}$$

$$= \sin 2x.\frac{1}{\sin y}.\cos y\frac{dy}{dx}$$

$$+\log\sin y.(2\cos 2x)$$

$$= \sin 2x.\cot y\frac{dy}{dx} + 2\cos 2x\log\sin y$$

$$\Rightarrow \frac{dy}{dx}\left(\frac{1}{y} - \sin 2x\cot y\right)$$

$$= 2\cos 2x\log\sin y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2\cos 2x\log\sin y}{\frac{1}{2}-\sin 2x\cot y}$$

(c) Given function is

$$f(x) = \begin{cases} \frac{1-e^{-x}}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

Differentiability at x=0

L.H.D.

$$= \lim_{x\to 0^-}\frac{f(x)+f(0)}{x-0}$$

$$= \lim_{h\to 0} -\frac{f(0-h)-f(0)}{0-h}$$

$$= \lim_{h\to 0}\frac{f(-h)-1}{-h}$$

$$= \lim_{h\to 0}\frac{\frac{1-e^h}{-h}-1}{-h}$$

$$= \lim_{h\to 0}\frac{1-e^h+h}{h^2} \quad (\text{form is } \frac{0}{0})$$

$$= \lim_{h\to 0}\frac{-e^h+1}{2h} \quad (\text{form is } \frac{0}{0})$$

$$= \lim_{h\to 0}\frac{-e^h}{2} = -\frac{1}{2}.$$

R.H.D.

$$= \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{h \to 0} \frac{f(0+h) - f(0)}{0+h}$$

$$= \lim_{h \to 0} \frac{f(h) - f(0)}{h}$$

$$= \lim_{h \to 0} \frac{\frac{1-e^{-h}}{h} - 1}{h}$$

$$= \lim_{h \to 0} \frac{1-e^{-h}-h}{h^2} \quad (\text{form is } \frac{0}{0})$$

$$= \lim_{h \to 0} \frac{e^{-h}-1}{2h} \quad (\text{form is } \frac{0}{0})$$

$$= \lim_{h \to 0} \frac{-e^{-h}}{2} = -\frac{1}{2}$$

At x=0, L.H.D. = R. H. D.

So the function is derivable at x=0. Since every derivable function is continuous so the function is continuous at x=0.

(d) The equation of the curve is

$$\sqrt{x} + \sqrt{y} = \sqrt{a} \quad \text{... (1)}$$

Let P be a point on the curve (1) whose coordinates are (x_1, y_1).

$$\therefore \sqrt{x_1} + \sqrt{y_1} = \sqrt{a} \quad \text{... (2)}$$

Differentiating both sides w.r.t. x, we get

$$\frac{d\sqrt{x}}{dx} + \frac{d\sqrt{y}}{dx} = \frac{d\sqrt{a}}{dx}$$

$$\Rightarrow \frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}.\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{\sqrt{y}}\frac{dy}{dx} = -\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

At the point (x_1, y_1), $\frac{dy}{dx} = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$

Equation of the targent to the curve (1) at the point (x_1, y_1) is

$$y - y_1 = \frac{dy}{dx}(x - x_1)$$

$$\Rightarrow y - y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1)$$

$$\Rightarrow y\sqrt{x_1} - y_1\sqrt{x_1} = -x\sqrt{y_1} + x_1\sqrt{y_1}$$

$$\Rightarrow x\sqrt{y_1} + y\sqrt{x_1} = x_1\sqrt{y_1} + y_1\sqrt{x_1}$$

$$\Rightarrow \sqrt{x_1}\sqrt{y_1}\ (\sqrt{x_1} + \sqrt{y_1})$$

$$\Rightarrow \sqrt{x_1}\sqrt{y_1}\sqrt{a}$$

$$\Rightarrow \frac{x}{\sqrt{x_1}\sqrt{a}} + \frac{y}{\sqrt{y_1}\sqrt{a}} = 1$$

Let the target at P intersect x-axis at A and y-axis at B.

$$\therefore 0A = \sqrt{x_1}\sqrt{a},\ 0B = \sqrt{b_1}\sqrt{a}$$

Sum of the intercepts $= OA + OB$

$$= \sqrt{x_1}\sqrt{a} + \sqrt{y_1}\sqrt{a}$$

$$= \sqrt{a}(\sqrt{x_1} + \sqrt{y_1}) = \sqrt{a}.\sqrt{a} = a.$$

Which is constant.

(e) Let $f(x) = 2\sin x + 3\tan x - 3x$

$$f'(x) = 2\cos x + 3\sec^2 x - 3$$

$$= 2\cos x + 3\tan^2 x > 0 \text{ for all } x \in \left(0, \frac{\pi}{2}\right)$$

So the function is increasing for $x \in \left(0, \frac{\pi}{2}\right)$

But $f(0) = 2\sin 0 + 3\tan 0 - 30$

$$\Rightarrow f(0) = 0$$

$$\therefore f(x) > f(0)$$

$$\Rightarrow 2\sin x + 3\tan x - 3x > 0$$

$$\Rightarrow 2\sin x + 3\tan x > 3x.$$

5. (a) Let $I=\int \frac{dx}{(1+x)\sqrt{1-x^2}}$

Let $1+x=\frac{1}{t}$

$\therefore dx=-\frac{1}{t^2}dt$

Also $x=\frac{1}{t}-1=\frac{1-t}{t}$

$1-x^2=1-\frac{(1-t)^2}{t^2}=\frac{t^2-(1-2t+t^2)}{t^2}=\frac{2t-1}{t^2}$

$\therefore \sqrt{1-x^2}=\frac{\sqrt{2t-1}}{t}$

$I=\int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}.\frac{\sqrt{2t-1}}{t}}=-\int \frac{dt}{\sqrt{2t-1}}$

Let $2t-1=u$

$\Rightarrow 2dt=du$

$\Rightarrow dt=\frac{1}{2}du$

$I=-\int \frac{\frac{1}{2}du}{\sqrt{u}}=-\frac{1}{2}\int u^{-\frac{1}{2}}du$

$=-\frac{1}{2}\frac{u^{1/2}}{1/2}+c=-\sqrt{u}+c=-\sqrt{2t-1}+c$

$=-\sqrt{\frac{2}{1+x}-1}+c=-\sqrt{\frac{1-x}{1+x}}+c$.

(b) Let $I=\int_0^1 \frac{\ln x}{\sqrt{1-x^2}}dx$

Let $x=\sin\theta$

$dx=\cos\theta\, d\theta$

When x = 0, $\theta=0$

When x = 1, $\theta=\frac{\pi}{2}$

$I=\int_0^{\pi/2}\frac{\ln\sin\theta}{\sqrt{1-\sin^2\theta}}.\cos\theta\, d\theta$

$=\int_0^{\pi/2}\frac{\ln\sin\theta}{\cos\theta}\cos\theta\, d\theta$

$=\int_0^{\pi/2}\ln\sin\theta\, d\theta$ ... (1)

Also $I=\int_0^{\pi/2}\ln\sin\left(\frac{\pi}{2}-\theta\right)d\theta$

$I=\int_0^{\pi/2}\ln\cos\theta\, d\theta$ ... (2)

Adding (1) and (2), we get

$2I=\int_0^{\pi/2}\ln\sin\theta\, d\theta+\int_0^{\pi/2}\ln\cos\theta\, d\theta$

$=\int_0^{\pi/2}(\ln\sin\theta+\ln\cos\theta)d\theta$

$=\int_0^{\pi/2}(\ln\sin\theta\cos\theta)d\theta$

$=\int_0^{\pi/2}\ln\left(\frac{\sin 2\theta}{2}\right)d\theta$

$=\int_0^{\pi/2}(\ln\sin 2\theta-\ln 2\, d\theta)$

$=\int_0^{\pi/2}\ln\sin 2\theta\, d\theta-\ln 2\int_0^{\pi/2}d\theta$

$=\int_0^{\pi/2}\ln\sin 2\theta\, d\theta-\frac{\pi}{2}\ln 2$ ... (3)

$=\int_0^{\pi/2}\ln\sin 2\theta\, d\theta=\int_0^{\pi}\ln\sin t\,.\frac{1}{2}dt$

$[2\theta=t,\ 2d\theta=dt\Rightarrow d\theta=\frac{1}{2}dt]$

$=\frac{1}{2}\int_0^{\pi}\ln\sin t\, dt$

$=\frac{1}{2}\left[\int_0^{\pi/2}\ln\sin t\, dt+\int_0^{\pi/2}\ln\sin(\pi-t)dt\right]$

$=\frac{1}{2}\left[\int_0^{\pi/2}\ln\sin t\, dt+\int_0^{\pi/2}\ln\sin t\, dt\right]$

$=\frac{1}{2}.2\int_0^{\pi/2}\ln\sin t\, dt$

$=\int_0^{\pi/2}\ln\sin\theta\, d\theta=I$.

From (3), we get

$2I=I-\frac{\pi}{2}\ln 2\Rightarrow I=-\frac{\pi}{2}\ln 2$.

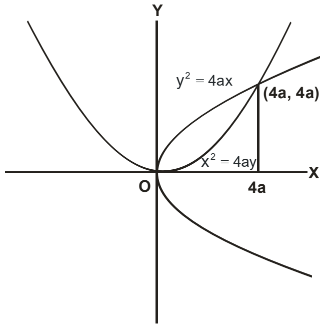
(c) The equations of two parabolas are

$y^2 = 4ax$ ... (1)

$x^2 = 4ax$ ... (2)

The points of intersection of (1) and (2) are (0, 0) and (4a, 4a).

Area enclosed between two curves



$$= \int_0^{4a} \sqrt{4ax\,dx} - \int_0^{4a} \frac{x^2}{4a} dx$$

$$= 2\sqrt{a}\int_0^{4a} x^{\frac{1}{2}} dx - \frac{1}{4a}\int_0^{4a} x^2\, dx$$

$$= 2\sqrt{a}\left[\frac{x^{3/2}}{3/2}\right]_0^{4a} - \frac{1}{4a}\left[\frac{x^3}{3}\right]_0^{4a}$$

$$= 2\sqrt{a}\,.\frac{2}{3}x\left[x\sqrt{x}\right]_0^{4a} - \frac{1}{12a}\left[x^3\right]_0^{4a}$$

$$= \frac{4\sqrt{a}}{3}\left[4a\sqrt{4a} - 0\right] - \frac{1}{12a}[64a^3 - 0]$$

$$= \frac{22}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2 \text{ sq. unit.}$$

(d) The general solution of the differential equation is

$y = a\sin t + b e^t$ ... (1)

$\frac{dy}{dx} = a\cos t + b e^t$ ... (2)

$\frac{d^2y}{dx^2} = -a\sin t + b e^t$ ... (3)

Adding (1) and (3), we get

$$y + \frac{d^2y}{dx^2 = 2b e^t}$$

$$\Rightarrow b e^t = \frac{1}{2}\left(y + \frac{d^2y}{dx^2}\right)$$

Subtracting (3) from (1), we get

$$y - \frac{d^2y}{dx^2} = 2a\sin t$$

$$\Rightarrow a = \frac{1}{2\sin t}\left(y + \frac{d^2y}{dx^2}\right)$$

From (2), we get

$$\frac{dy}{dx} = \frac{1}{2\sin t}\left(y - \frac{d^2y}{dt^2}\right)\cos t + \frac{1}{2}\left(b + \frac{d^2y}{dt^2}\right)$$

Which is the required equation.

(e) Given differential equation is

$$(1 + y^2)dx + (x - e^{-\tan^{-1}y})dy = 0$$

$$\Rightarrow (1 + y^2)\frac{dx}{dy} + x - e^{-\tan^{-1}y} = 0$$

$$\Rightarrow (1 + y^2)\frac{dx}{dy} + x = e^{-\tan^{-1}y}$$

$$\Rightarrow \frac{dx}{dy} + \frac{1}{1+y^2}x = \frac{1}{1+y^2}e^{-\tan^{-1}y} \quad \text{... (1)}$$

$$P = \frac{1}{1+y^2}$$

Integrating factor

$$= e^{\int p\,dy} = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$$

Multiplying both sides of (1) by $e^{\tan^{-1}}y$, we get

$$e^{\tan^{-1}}y\,.\,\frac{dx}{dt} + \frac{1}{1+y^2}e^{\tan^{-1}y}.x$$

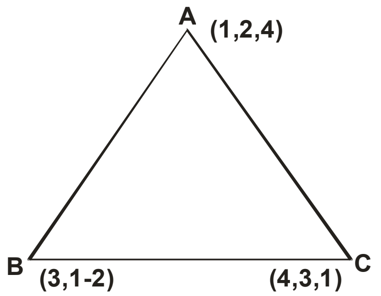
$$= \frac{1}{1+y^2}e^{\tan^{-1}y'}.e^{-\tan^{-1}y}$$

$$\Rightarrow \frac{d}{dy}(x\, e^{\tan^{-`}y}) = \frac{1}{1+y^2}$$

Integrating both sides, we get

$$x e^{\tan^{-1}y} = \int \frac{1}{1+y^2}dy.$$

6. (a) Let ABC be the triangle. Let the coordinates of A, B and C be (1,2,4), (3,1,-2) and (4,3,1) respectively.



$\overrightarrow{BC} =$ P.V. of C - P. V. of B

$= (4\hat{i} + 3\hat{j} + \hat{k}) - (3\hat{i} + \hat{j} + 2\hat{k})$

$= \hat{i} + 2\hat{j} + 3\hat{k}$

$\overrightarrow{BA} = \text{P.V.of A} - \text{P.V.of B}$

$= \hat{i} + 2\hat{j} + 4\hat{k} - (3\hat{i} + \hat{j} - 2\hat{k})$

$= -2\hat{i} + \hat{j} + 6\hat{k}$

Vector area of the $\Delta ABC = \frac{1}{2}(\overrightarrow{BC} \times \overrightarrow{BA})$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -2 & 1 & 6 \end{vmatrix}$$

$= \frac{1}{2}[\hat{i}(12-3) - \hat{j}(6+6) + \hat{k}(1+4)$

$= \frac{1}{2}(8\hat{i} - 12\hat{j} + 5\hat{k})$

Area of $\Delta ABC = \frac{1}{2}\sqrt{8^2 + 12^2 + 5^2}$

(b) To prove that

$[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]$

L.H.S.

$= [\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}]$

$= (\vec{a}+\vec{b}).[(\vec{b}+\vec{c}) \times (\vec{c}+\vec{a})]$

$= (\vec{a}+\vec{b}).[(\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{c}\times\vec{c} + \vec{c}\times\vec{a}]$

$= (\vec{a}+\vec{b}).[\vec{b}\times\vec{c} + \vec{b}\times\vec{a} + \vec{0} + \vec{c}\times\vec{a}]$

$= \vec{a}.(\vec{b}\times\vec{c}) + \vec{a}.(\vec{b}\times\vec{a}) + \vec{a}.(\vec{c}\times\vec{a}) +$

$\vec{b}.(\vec{b}\times\vec{a}) + \vec{b}.(\vec{b}\times\vec{c})$

$= \vec{a}.(\vec{b}\times\vec{c}) + 0 + 0 + 0 + \vec{a}.(\vec{b}\times\vec{c})$

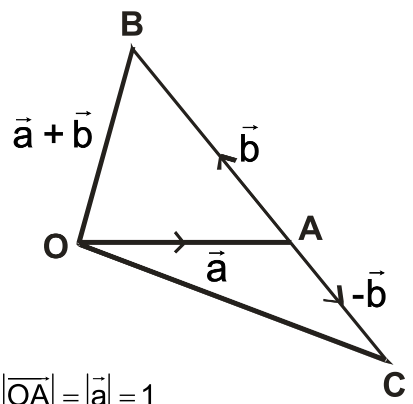
$= 2[\vec{a}.(\vec{b}\times\vec{c})]$

$= 2[\vec{a}\vec{b}\vec{c}]$

(e) Let $\overline{OA} = \vec{a}$, $\overline{AB} = \vec{b}$

$\therefore \overline{OB} = \overline{OA} + \overline{AB} = \vec{a} + \vec{b}$

Given that



$|\overline{OA}| = |\vec{a}| = 1$

$|\overline{AB}| = |\vec{b}| = 1$

$|\overline{OB}| = |\vec{a}+\vec{b}| = 1$

Let us produce BA to C such that BA=AC.

$\because \overline{AB} = \vec{b} \Rightarrow \overline{BA} = -\vec{b}$

$\therefore \overline{AC} = -\vec{b}$

$\overline{OC} = \overline{OA} + \overline{AC} = \vec{a} + (-\vec{b}) = \vec{a} - \vec{b}$

Since $OA = AB = OB$, $\angle AOB = 60^0$

$\therefore \angle AOC = \angle ACO = 30^0$

$\therefore \angle BOC = 60^0 + 30^0 = 90^0$

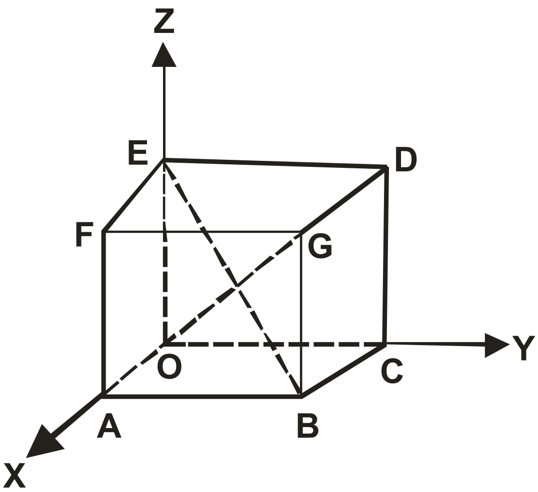
From the ΔOBC, $\tan\angle OBC = \frac{OC}{OB}$

$$\Rightarrow \tan 60^0 = \frac{|\vec{a}-\vec{b}|}{|\vec{a}+\vec{b}|}$$

$$\Rightarrow \sqrt{3} = \frac{|\vec{a}-\vec{b}|}{1}$$

$$\Rightarrow |\vec{a}-\vec{b}| = \sqrt{3}.$$

(d) Let OABCD be the cube, the length of whose edges are a each. Let us take O as origin, OA along x-axis, OC along y-axis and OE along z-axis. The coordinates of the vertices are O (o,o,o) A(a,o,a), B(a,a,o), C(o,a,o), D(o,a,a), E(o,o,a), F(a,o,a) and G (a,a,a).



OG and EB are two main diagonals of the cube.

The d.rs of OG are <a-o, a-o, a-o)=<a,a,a>

The d.rs of EB are two main diagonals of the cube.

The d.rs of OG are <a-o, a-o, a-o) = <a,a,a>

Let θ be the angle between them.

$$\text{Cos}\,\theta = \frac{a.a + a.a + a(-a)}{\sqrt{a^2+a^2+a^2}\sqrt{a^2+a^2+(-a)^2}}$$

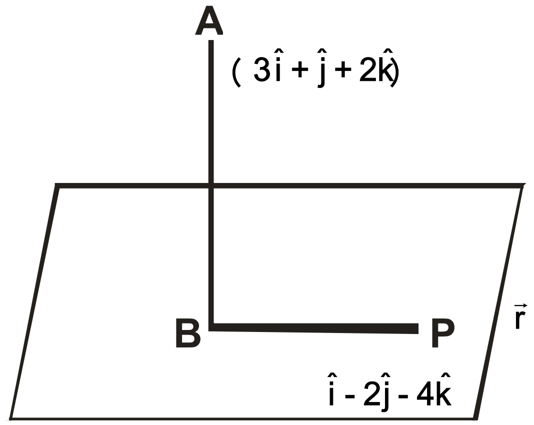
$$= \frac{a^2+a^2-a^2}{\sqrt{3a^2}\sqrt{3a^2}} = \frac{a^2}{\sqrt{3a}.\sqrt{3a}} = \frac{1}{3}$$

$$\theta = \text{Cos}^{-1}\left(\frac{1}{3}\right).$$

$\therefore$ The angle between the two main diagonals of a is $\text{Cos}^{-1}\frac{1}{3}$.

(e) Let A and B are two points whose P.V.s are $3\hat{i}+\hat{j}+2\hat{k}$ and $\hat{i}-2\hat{j}-4\hat{k}$ respectively.

Let P be any point on the plane whose position vector is $\hat{r}$.



$\overrightarrow{BA} =$ P.V. of A - P.V. of B.

$= (3\hat{i}+\hat{j}+2\hat{k}) - (\hat{i}-2\hat{j}-4\hat{k})$

$= 2\hat{i}+3\hat{j}+6\hat{k}$.

Since the plane is perpendicular to AB, so $\overrightarrow{AB}$ is perpendicular to $\overrightarrow{BP}$.

$\therefore \overrightarrow{BP} \,.\, \overrightarrow{AB} = 0$

$$\Rightarrow \left[\vec{r} - (\hat{i}-2\hat{j}-4\hat{k})\right].\,(2\hat{i}+3\hat{j}+6\hat{k}) = 0$$

$$\Rightarrow \vec{r}.(2\hat{i}+3\hat{j}+6\hat{k}) = (\hat{i}-2\hat{j}-4\hat{k}).(2\hat{i}+3\hat{j}+6\hat{k})$$

$$\Rightarrow \vec{r}.(2\hat{i}+3\hat{j}+6\hat{k}) = 2-6-24$$

$$\Rightarrow \vec{r}.(2\hat{i}+3\hat{j}+6\hat{k}) = -28$$

GROUP - C

7. (a) Two given functions are

$f: X \to Y$ and $g: Y \to Z$

First we shall show that if f and g are invertible then gof is also invertible i.e. if f and g are one-one and on oto them gof is also one-one and on to.

(i) Let f and g are one-one let - $x_1, x_2 \in X$

$\therefore$ (gof) (x_1) = (gof) (x_2)

$\Rightarrow g[(x_1)] = g[f(x_2)]$

$\Rightarrow f(x_1) = f(x_2)$ ($\because$ g is one-one)

$\Rightarrow x_1 = x_2$ ($\because$ f is one-one)

Thus for $x_1, x_2 \in X$,

$(gof)(x_1) = (gof)(x_2) \Rightarrow x_1 = x_2$

So gof is one-one

(ii) Let f and g be two on to functions.

Let $z \in Z$

Given that $g: Y \to Z$ be an on to function so there exists an element $y \to Y$ such that g(y) = Z.

Since $f: X \to Y$ be on to, there exists an element $x \in X$ such that f(x) = y.

Now g(y) = Z

$\Rightarrow g(f(x)) = Z$

$\Rightarrow (gof)(x) = Z$

Thus for any element $z \in Z$, there exists $x \in X$ such that (gof) (x) = z.

$\Rightarrow$ gof : $X \to Z$ is an on to function.

Thus we see that if f and g are one-one and onto so gof is one-one and onto.

$\Rightarrow$ gof is invertible.

$\Rightarrow$ $(gof)^{-1}$ exists.

Here $f: X \to Y$ is objective.

There exists $y \in Y$ such that $f(x) = y$

$\Rightarrow x = 'f'(y)$ ... (1)

Again given that $g: Y \to Z$ is objective.

So there exists an element $z \in Z$ such that g(y)=z.

$\Rightarrow y = g^{-1}(z)$ ...(2)

$\therefore (gof)x = g[f(x)] = g(y) = z$

$\Rightarrow x = (gof)^{-1}(Z)$ ...(3)

Also $x = f^{-1}(y) = f^{-1}\left[g^{-1}(z)\right]$

$= (f^{-1} o g^{-1})(z)$ ...(4)

From (3) and (4), we see that

$(gof)^{-1} = f^{-1} o g^{-1}$.

(b) ABC is a right angled triangle at A.

$\therefore < A = 90^0$

$\Rightarrow a^2 = b^2 + c^2$

L.H.S. = $\tan^{-1}\frac{b}{a+c} + \tan^{-1}\frac{c}{a+b}$

$$= \tan^{-1}\left(\frac{\frac{b}{a+c} + \frac{c}{a+b}}{1 - \frac{b}{a+c}.\frac{c}{a+b}}\right)$$

$$= \tan^{-1}\left[\frac{b(a+b) + c(a+c)}{(a+c)(a+b) - bc}\right]$$

$$= \tan^{-1}\left[\frac{ab + b^2 + ac + c^2}{a^2 + ab + ac + bc - bc}\right]$$

$$= \tan^{-1}\left(\frac{ab + ac + a^2}{a^2 + ab + ac}\right)$$

$$= \tan^{-1} 1 = \frac{\pi}{4}.$$

(c) The given L.P.P. is

Maximize $Z = 3x_1 + 2x_2$ ... (1)

Subject to $-2x_1 + x_2 \leq 1$ ... (2)

$x_1 \leq 2$ ... (3)

$x_1 + x_2 \leq 3$ ... (4)

$x_1, x_2 \geq 0$ ... (5)

The feasible region is in the first quadrant, changing the inequations to equations, we get

$-2x_1 + x_2 = 1$ ... (6)

$x_1 = 2$ ... (7)

$x_1 + x_2 = 3$ ... (8)

$x_1 = 0,\ x_2 = 0$ ... (9)

From (6), we get

when $x_1 = 0,\ x_2 = 1$

when $x_2 = 0,\ x_1 = -\frac{1}{2} = -0.5$

This is show in the table below.

x_1	0	–0.5
x_2	1	0

Putting (0, 0) in the in equations (2), we get $0 \le 1$

$\Rightarrow$ The half plane is towards the origin.

The line (7) is parallel to x_2-axis.

From (8), we get

When x_2=0, then x_1=3

When x_1=0, then x_2=3

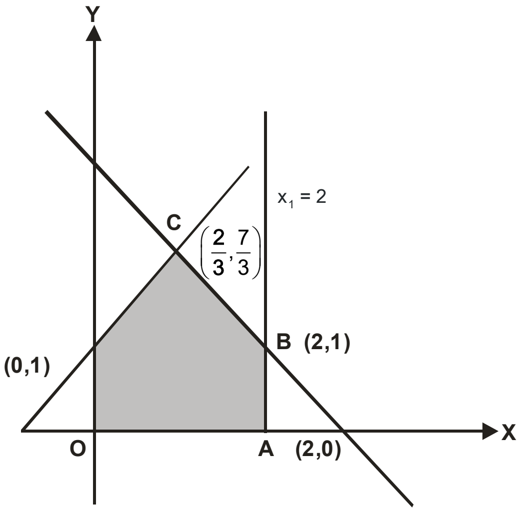
This is shown in the table below.

x_1	3	0
x_2	0	3

The line (8) intersects axes at (3,0) and (0,3).

Putting (0, 0) in the equation (4),

we get 0+0$\le$3.



The half plane is towards the origin.

Here O ABCD is the feasible region

Where O is (0,0), A (2,0), B is (2,1), $C\left(\frac{2}{3},\frac{7}{3}\right)$ and D is (0, 1).

The value of Z at the above corner points are given below :

Point	x_1	x_2	$Z = 3x_1+2x_2$
O	0	0	Z = 0
A	2	0	Z = 3.2 + 2.0 = 6
B	2	1	Z = 3.2 + 2.1 = 8
C	$\frac{2}{3}$	$\frac{7}{3}$	$Z = 3.\frac{2}{3} + 2.\frac{7}{3} = \frac{20}{3}$
D	0	1	Z = 3.0 + 2.1 = 2

The maximum value of Z is 8.

It is obtained when $x_1 = \frac{2}{3},\ x_2 = \frac{7}{3}$.

8. (a) The given linear equations are

$x + y + z = 4$

$2x - y + 3z = 1$

$3x + 2y - z = 1$

$$\Rightarrow \begin{bmatrix} x+y+z \\ 2x-y+3z \\ 3x+2y-z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow AX = B$

Where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{bmatrix},\ X = \begin{bmatrix} x \\ y \\ z \end{bmatrix},\ B = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$\Rightarrow X = A^{-1}B$

$$\Rightarrow X = \frac{1}{|A|}(\text{adj}\,A)B \qquad \text{... (1)}$$

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & -1 & 3 \\ 3 & 2 & -1 \end{vmatrix}$$

$= 1(1-6) - 1(-2-9) + 1(4+3)$

$= -5 + 11 + 7 = 13 \neq 0$.

C_{11} = cofactor of

$1 = (-1)^{1+1}\begin{vmatrix} -1 & 3 \\ 2 & -1 \end{vmatrix} = 1 - 6 = -5$

$C_{12} = (-1)^{1+2}\begin{vmatrix} 2 & 3 \\ 3 & -1 \end{vmatrix} = -(-2-9) = 11$

$C_{13} = (-1)^{1+3}\begin{vmatrix} 2 & -1 \\ 3 & 2 \end{vmatrix} = 4 - (-3) = 7$

$C_{21} = (-1)^{2+1}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -(-1-2) = 3$

$C_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 1 \\ 3 & -1 \end{vmatrix} = -1 - 3 = -4$

$C_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 1 \\ 3 & 2 \end{vmatrix} = -(2-3) = 1$

$C_{31} = (-1)^{3+1}\begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} = 3 - (-1) = 4$

$C_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 1 \\ 2 & 3 \end{vmatrix} = -(3-2) = -1$

$C_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = -1 - 2 = -3$

Matrix of cofactors

$$= \begin{bmatrix} -5 & 11 & 7 \\ 3 & -4 & 1 \\ 4 & -1 & -3 \end{bmatrix}$$

$$\text{Adj } A = \begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}$$

From (1), we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -5 & 3 & 4 \\ 11 & -4 & -1 \\ 7 & 1 & -3 \end{bmatrix}\begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix}$$

$$= \frac{1}{13}\begin{bmatrix} -20+3+4 \\ 44-4-1 \\ 28+1-3 \end{bmatrix} = \frac{1}{13}\begin{bmatrix} -13 \\ 39 \\ 26 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ 3 \\ 2 \end{bmatrix}$$

$\therefore x = -1,\ y = 3,\ z = 2$.

(b) There is a well shuffled deck of 52 cards. Two cards are drawn without replacement.

Let X denotes the number of aces in a successive draw of two cards without replacement.

X is a random variable which bases values 0, 1 or 2. We draw cards without replacement.

P(X = 0) = P (no ace and no ace)

$= \frac{48}{52} \cdot \frac{47}{51}$.

P(X=1) =P[(one ace and no ace) or (no ace and one ace)]

= P (one ace and no ace) +

P (no ace and one ace)

$= \frac{4}{52} \cdot \frac{48}{51} + \frac{48}{52} \cdot \frac{4}{51} = \frac{16}{221}$.

P(X=2) = P (one ace and one ace)

= P (one ace) x P (one ace)

$= \frac{4}{52} \cdot \frac{3}{51} = \frac{1}{221}$

The probability distribution of X is given in the following table.

X = x	0	1	2
P(x)	$\frac{196}{221}$	$\frac{16}{221}$	$\frac{1}{221}$

The mean $\bar{x}$ is given by

$\bar{x} = 0 \text{ x } \frac{196}{221} + 1 \text{ x } \frac{16}{221} + 2 \text{ x } \frac{1}{221} = \frac{18}{221}$.

The variance $\sigma^2 = \sum_{i=0}^{2} xi^2\, P(xi) - \bar{x}^2$

$= o^2 \text{ x } \frac{196}{221} + 1^2 \text{ x } \frac{16}{221} + 2^2 \text{ x } \frac{1}{221} - \left(\frac{18}{221}\right)^2$

$= \frac{20}{221} - \left(\frac{18}{221}\right)^2$

$= \frac{20}{221} - \left(\frac{18}{221}\right)^2$.

(c) Given $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix}$

We know A = IA

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} A$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$(R_2 \to R_2 - R_1, R_3 \to R_3 - R_1)$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} A$$

$(R_3 \to R_3 - R_2)$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & -1 & 1 \end{bmatrix} A$$

$(R_2 \to -\frac{1}{2} R_2)$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 0 & -1 & 1 \end{bmatrix} A$$

$(R_2 \to R_2 + \frac{1}{2} R_3)$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 1 & -1 & 1 \end{bmatrix} A$$

$(R_1 \to R_1 - R_2)$

Thus $A^{-1} \begin{bmatrix} \frac{1}{2} & 1 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \\ 1 & -1 & 1 \end{bmatrix}$

9. (a) Given that

$$x = \frac{1-\text{Cos}^2\theta}{\text{Cos}\,\theta},\ y = \frac{1-\text{Cos}^{2n}\theta}{\text{Cos}^n\theta}$$

$$\Rightarrow x = \frac{1}{\text{Cos}\,\theta} - \text{Cos}\,\theta,\ y = \frac{1}{\text{Cos}^n\theta} - \text{Cos}^n\theta$$

$$\Rightarrow x = \text{Sec}\,\theta - \text{Cos}\,\theta,\ y = \text{Sec}^n\theta - \text{Cos}^n\theta.$$

$$\frac{dx}{d\theta} = \frac{d(\text{Sec}\theta - \text{Cos}\theta)}{d\theta} = \frac{d\text{Sec}\theta}{d\theta} - \frac{d\text{Cos}\theta}{d\theta}$$

$$= \text{Sec}\,\theta.\tan\theta - (-\text{Sin}\,\theta)$$

$$= \text{Sec}\,\theta.\tan\theta + \tan\theta.\text{Cos}\,\theta$$

$$= \tan\theta(\text{Sec}\theta + \text{Cos}\theta)$$

$$y = \text{Sec}^n\theta - \text{Cos}^n\theta$$

$$\frac{dy}{d\theta} = \frac{d\text{Sec}^n\theta}{d\text{Sec}\theta} - \frac{d\text{Cos}^n\theta}{d\theta}$$

$$= \frac{d\text{Sec}^n\theta}{d\text{Sec}\theta}.\frac{d\text{Sec}\theta}{d\theta} - \frac{d\text{Cos}^n\theta}{d\text{Cos}\theta}.\frac{d\text{Cos}\theta}{d\theta}$$

$$= n\text{Sec}^{n-1}\theta.\text{Sec}\theta\tan\theta - n\text{Cos}^{n-1}\theta.(-\text{Sin}\theta)$$

$$= n\tan\theta(\text{Sec}^n + \text{Cos}^n\theta)$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{n\tan\theta(\text{Sec}^n\theta + \text{Cos}^n\theta)}{\tan\theta(\text{Sec}\theta + \text{Cos}\theta)}$$

$$= n\frac{(\text{Sec}^n\theta + \text{Cos}^n\theta)}{\text{Sec}\theta + \text{Cos}\theta}$$

$$\left(\frac{dy}{dx}\right)^2 = n^2\frac{(\text{Sec}^n\theta + \text{Cos}^n\theta)^2}{(\text{Sec}\theta + \text{Cos}\theta)^2}$$

$$= n^2\frac{\left[(\text{Sec}^n\theta - \text{Cos}^n\theta)^2 + 4\text{Sec}^n\theta.\text{Cos}^n\theta\right]}{(\text{Sec}\,\theta - \text{Cos}\theta)^2 + 4\,\text{Sec}\theta.\text{Cos}\theta}$$

$$= n^2\left(\frac{y^2+4}{x^2+4}\right)$$

(b) The equation of the curve is

$$x^2y - x + y = 0$$

$$\Rightarrow y(x^2+1) = x$$

$$\Rightarrow y = \frac{x}{x^2+1}$$

Slope of the tangent is given by

$$\frac{dy}{dx}=\frac{d}{dx}\left(\frac{x}{x^2+1}\right)$$

$$=\frac{(x^2+1)\frac{dx}{dx}-x\frac{d(x^2+1)}{dx}}{(x^2+1)^2}$$

$$=\frac{x^2+1-x.2x}{(x^2+1)^2}$$

$$=\frac{1-x^2}{(x^2+1)^2}=s(say)$$

$$\therefore s=\frac{1-x^2}{(x^2+1)^2}$$

$$\frac{ds}{dx}=\frac{(x^2+1)^2(-2x)-(1-x^2).2(x^2+1).2x}{(x^2+1)^4}$$

$$=\frac{-2x(x^2+1)-4x(1-x^2)}{(x^2+1)^3}$$

$$=\frac{2x^3-6x}{(x^2+1)^3}$$

$$\frac{d^2s}{dx^2}=\frac{(x^2+1)^3.(6x^2-6)-(2x^3-6x)3(x^2+1)^2.2x}{(x^2+1)^6}$$

$$=\frac{(x^2+1)(6x^2-6)-6x(2x^3-6x)}{(x^2+1)^4}$$

For maximum or minimum value of s,

$$\frac{ds}{dx}=0$$

$$\Rightarrow\frac{2x^3-6x}{(x^2+1)^3}=0$$

$$\Rightarrow 2x(x^2-3)=0$$

$$\Rightarrow x=0,\pm\sqrt{3}.$$

When $x=0,\ \frac{d^2s}{dx^2}$ is -ve.

$\therefore$ s is maximum when $x=0,\ y=0$.

The slope of the tangent is maximum at (0,0).

When $x=\pm\sqrt{3},\ \frac{d^2s}{dx^2}$ is +ve

$\Rightarrow$ s is minimum when $x=\pm\sqrt{3}$.

10. (a) Let $I=\int\frac{2\cos x+7}{4-\sin x}dx$

$$=2\int\frac{\cos x}{4-\sin x}dx+7\int\frac{1}{4-\sin x}dx$$

$$=I_1+I_2\ (say).$$

$$I_1=2\int\frac{\cos x}{4-\sin x}dx$$

$$=2\int-\frac{dt}{t}$$ (4-sin x=t

= -2 ln t+c - cosx dx=dt

=-2 ln (4-sin x)+c $\Rightarrow$ cos x dx=-dt)

$$I_2=7\int\frac{1}{4-\sin x}dx$$

$$=7\int\frac{1}{4-\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}dx$$

$$=7\int\frac{1+\tan^2\frac{x}{2}}{4+4\tan^2\frac{x}{2}-2\tan\frac{x}{2}}dx$$

$$=7\int\frac{\sec^2\frac{x}{2}dx}{4\tan^2\frac{x}{2}-2\tan\frac{x}{2}+4}$$

$$=7\int\frac{\sec^2\frac{x}{2}dx}{4\tan^2\frac{x}{2}-\frac{1}{2}\tan\frac{x}{2}+1}$$

$[\tan\frac{x}{2}=t$

$\therefore\frac{1}{2}\sec^2\frac{x}{2}dx=dt$

$\Rightarrow\sec^2\frac{x}{2}dx=2dt]$

$$=\frac{7}{4}\int\frac{2dt}{t^2-\frac{1}{2}t+1}$$

$$=\frac{7}{2}\int\frac{dt}{t^2-2.t.\frac{1}{4}+\frac{1}{16}+1-\frac{1}{16}}$$

$$=\frac{7}{2}\int\frac{dt}{\left(t-\frac{1}{4}\right)^2+\left(\frac{\sqrt{15}}{4}\right)^2}$$

$$=\frac{7}{2}.\frac{1}{\frac{\sqrt{15}}{4}}\tan^{-1}\frac{t-\frac{1}{4}}{\frac{\sqrt{15}}{4}}+c$$

$$=\frac{14}{\sqrt{15}}\tan^{-1}\left[\frac{4\tan\frac{x}{2}-1}{\sqrt{15}}\right]+c$$

From (1), we get

$I=I_1+I_2$

$$=-2\ln(4-\sin x)+\frac{14}{\sqrt{15}}\tan^{-1}\left[\frac{4\tan\frac{x}{2}-1}{\sqrt{15}}\right]+c$$

(b) The given differential equation is

$(4x+6y+5)dx-(2x+3y+4)dy=0$

$\Rightarrow(2x+3y+4)dy=(4x+6y+5)dx$

$$\Rightarrow\frac{dy}{dx}=\frac{2(2x+3y)+5}{2x+3y+4} \quad ...(1)$$

Let $2x+3y=v$

$$\therefore 2+3\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow 3\frac{dy}{dx}=\frac{dv}{dx}-2$$

$$\Rightarrow\frac{dy}{dx}=\frac{1}{3}\left(\frac{dv}{dx}-2\right)$$

From (1), we get

$$\frac{1}{3}\left(\frac{dv}{dx}-2\right)=\frac{2v+5}{v+4}$$

$$\Rightarrow\frac{dv}{dx}-2=\frac{6v+15}{v+4}$$

$$\Rightarrow\frac{dv}{dx}=2+\frac{6v+15}{v+4}$$

$$=\frac{2v+8+6v+15}{v+4}$$

$$=\frac{8v+23}{v+4}$$

$$\Rightarrow\frac{v+4}{8v+23}dv=dx$$

Integrating both sides, we get

$$\int\frac{v+4}{8v+23}dv=\int dx$$

$$\Rightarrow\frac{1}{8}\int\frac{8v+32}{8v+23}dv=x+c$$

$$\Rightarrow\frac{1}{8}\int\frac{(8v+23)+9}{8v+23}dv=x+c$$

$$\Rightarrow\frac{1}{8}\int 1+\frac{9}{8v+23}dv=x+c$$

$$\Rightarrow\frac{1}{8}v+\frac{9}{8}.\frac{1}{8}\int\frac{8dv}{8v+23}=x+c$$

$$\Rightarrow\frac{1}{8}v+\frac{9}{64}\ln(8v+23)=x+c$$

$$\Rightarrow\frac{1}{8}(2x+3y)\frac{9}{64}\ln\left[8(2x+3y)+23\right]=x+c$$

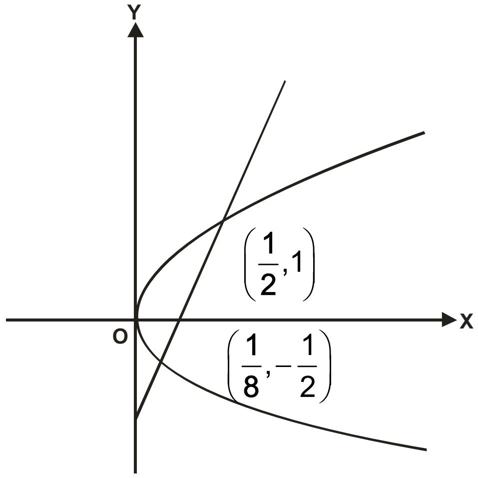
(c) The equation of two curves are

$y^2=2x$ ... (1)

$y=4x-1$ ... (2)

The intersection of (1) & (2) are

$\left(\frac{1}{8},\frac{-1}{2}\right)$ and $\left(\frac{1}{2},1\right)$



Required area

$$=\int_{\frac{1}{8}}^{\frac{1}{2}}\sqrt{2x}\ dx-\int_{\frac{1}{8}}^{\frac{1}{2}}(4x-1)dx$$

$$=\sqrt{2}\int \sqrt{x}\,dx-4\int_{\frac{1}{8}}^{\frac{1}{2}}x\,dx+\int_{\frac{1}{8}}^{\frac{1}{2}}dx$$

$$=\sqrt{2}.\left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]^{\frac{1}{2}}-4\left[\frac{x^2}{2}\right]^{\frac{1}{2}}+[x]_{\frac{1}{8}}^{\frac{1}{2}}$$

$$=\frac{2\sqrt{2}}{3}\left[x\sqrt{x}\right]_{\frac{1}{8}}^{\frac{1}{2}}-2\left[x^2\right]_{\frac{1}{8}}^{\frac{1}{2}}+\left(\frac{1}{2}+\frac{1}{8}\right)$$

$$=\frac{2\sqrt{2}}{3}\left[\frac{1}{2}.\frac{1}{\sqrt{2}}-\frac{1}{8}.\frac{1}{2\sqrt{2}}\right]-2\left[\frac{1}{4}-\frac{1}{64}\right]+\frac{3}{8}$$

$$=\frac{2\sqrt{2}}{3}\left[\frac{1}{2\sqrt{2}}-\frac{1}{16\sqrt{2}}\right]-2\ .\ \frac{15}{64}+\frac{3}{8}$$

$$=\frac{2\sqrt{2}}{3}.\frac{7}{16\sqrt{2}}-\frac{15}{32}+\frac{3}{8}$$

$$=\frac{7}{4}-\frac{15}{32}+\frac{3}{8}=\frac{28-45+72}{96}$$

$$=\frac{100-45}{96}=\frac{65}{96}\ \text{sq. unit.}$$

11. (a) Given that

$$\vec{a}=2\hat{i}+\hat{k}$$

$$\vec{b}=\hat{i}+\hat{j}+\hat{k}$$

$$\vec{c}=4\hat{i}-3\hat{j}+7\hat{k}$$

Let $\vec{r}=x\hat{i}+y\hat{j}+z\hat{k}$

Given that $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$

$$\Rightarrow\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\x&y&z\\1&1&1\end{vmatrix}=\begin{vmatrix}\hat{i}&\hat{j}&\hat{k}\\4&-3&7\\1&1&1\end{vmatrix}$$

$$\Rightarrow\hat{i}(y-z)-\hat{j}(x-z)+\hat{k}(x-y)$$

$$=\hat{i}(-3-7)-\hat{j}(4-7)+\hat{k}(4+3)$$

$$\Rightarrow\hat{i}(y-z)-\hat{j}(z-x)+\hat{k}(x-y)$$

$$=-10\hat{i}+3\hat{j}+7\hat{k}$$

$$\Rightarrow y-z=-10 \qquad ...(1)$$

$$z-x=3 \qquad ...(2)$$

$$x-y=7 \qquad ...(3)$$

Also given that $\vec{r}\ .\ \vec{a}=0$

Solving (1), (2), (3) and (4), we get

$x=-1,\ y=-8,\ z=2.$

(b) Two given lines are

$$\frac{x-3}{3}=\frac{y-8}{1}=\frac{z-3}{1} \qquad ...(1)$$

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4} \qquad ...(2)$$

The shortest distance between the two lines (1) and (2) is

$$\text{S.D.}=\frac{\begin{vmatrix}x_2-x_1&y_2-y_1&z_2-z_1\\l_1&m_1&n_1\\l_2&m_2&n_2\end{vmatrix}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}}$$

$$\begin{vmatrix}x_2-x_1&y_2-y_1&z_2-z_1\\l_1&m_1&n_1\\l_2&m_2&n_2\end{vmatrix}$$

$$=\begin{vmatrix} -3-3 & -7-8 & 6-3 \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$=\begin{vmatrix} -6 & -15 & 3 \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$=-6(-4-2)-(-15)(12+3)+3(6-3)$$

$$=36+225+9=270.$$

$$\sqrt{(m_1n_2-m_2n_1)^2+(n_1l_2-n_2l_1)^2+(l_1m_2-l_2m_1)^2}$$

$$=\sqrt{(-4-2)^2+(-3-12)^2+(6-3)^2}$$

$$=\sqrt{(-6)^2+(-15)^2+(3)^2}$$

$$=\sqrt{36+225+9}=\sqrt{270}$$

Required shortest distance

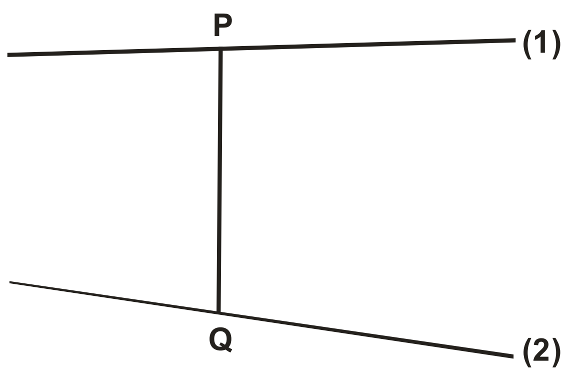
$$=\frac{270}{\sqrt{270}}=\sqrt{270}.$$

Here two given lines are

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}=\alpha\,(\text{say})$$

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}=\beta\,(\text{say})$$

Let PQ be the shortest distance.



Any point on the line(1) is

$(3\alpha+3,-\alpha+8,\alpha+3)$

The coordinates of P are also

$(3\alpha+3,-\alpha+8,\alpha+3)$ for same value of α.

The point on the line (2) is

$(-3\beta-3,2\beta-7,4\beta+6)$

The coordinates of Q are also

$(-3\beta-3,2\beta-7,4\beta+6)$

The d.rs of PQ are

$< 3\alpha+3\beta+6,\ -\alpha-2\beta+15,\ \alpha-4\beta-3 >$

D.rs of the given lines are

$< 3,-1,1 >$ and $< -3,2,4 >$.

Since PQ is perpendicular to the given lines, we have

$$3(6+3\alpha+3\beta)-(15-\alpha-2\beta)$$
$$+1.(-3+\alpha-4\beta)=0$$

and $-3(6+3\alpha+3\beta)+2(15-\alpha-2\beta)$
$$+4(-3+\alpha-4\beta)=0$$

$$\Rightarrow 18+9\alpha+9\beta-15+\alpha+2\beta$$
$$-12+4\alpha-16\beta=0$$

and $-18-9\alpha-9\beta+30-2\alpha-4\beta$
$$-12+4\alpha-16\beta=0$$

$$\Rightarrow 11\alpha+7\beta=0$$

and $-7\alpha-29\beta=0$

$$\Rightarrow \alpha=0,\ \beta=0$$

The coordinates of P and Q are

(3,8,3) and (-3,-7,6)

Equation of the line of shortest distance is

$$\frac{x-3}{-3-3}=\frac{y-8}{-7-8}=\frac{z-3}{6-3}$$

$$\Rightarrow \frac{x-3}{-6}=\frac{y-8}{-15}=\frac{z-3}{3}.$$

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2017 (A)

Full Marks : 100 | Time : 3 hours

Answer all questions as per instructions given in each.
The figures in the right-hand margin indicate marks.
Electronic gadgets are not allowed in the Examination Hall

GROUP - A

(Marks : 10)

1. Answer all questions : [1x10=10]

(a) Write the minimum value of n such that $\frac{d^n(3x^3+7)^{15}}{dx^n}=0$

(b) Write the interval in which the function $\sin^2 x - x$ is increasing.

(c) Write the value of $\int_0^1 \{x\}\,dx$ where $\{x\}$ stands for fractional part of x.

(d) Write the order of the differential equation of the family of circles $ax^2+ay^2+2gx+2fy+c=0$.

(e) If the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ form the slides $\overline{BC}, \overline{CA}$ and $\overline{AB}$ respectively of a triangle ABC, then write the value of $\vec{a} \times \vec{c} + \vec{b} \times \vec{c}$.

(f) Write the equation of the plane meeting the coordinate axes in A, B and C in order given that (a, b, c) is the centroid of Δ ABC.

(g) Find the value of k such that the line $\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2}$ lies on the plane $2x+4y+z=7$.

(h) If I_n is an identity matrix of order n and k being a natural number, then write the matrix I_n^k.

(i) Write the number of ways in which 5 boys and 5 girls can set around a round table.

(j) One card is drawn from a pack of 52 cards. Write the probability that the card drawn is either a king or a spade.

GROUP - B

(Marks : 60)

2. Answer any five questions : [3x5=15]

(a) Differentiate $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$.

(b) If $y=x^4e^{2x}$, then find y_n.

(c) If $y=x+\cfrac{1}{x+\cfrac{1}{x+\ldots\infty}}$, Find $\frac{dy}{dx}$, then r.h.s. being a valid expression.

(d) If $u=\ln(x^3+y^3+z^3-3xyz)$ then show that $\frac{du}{dx}+\frac{du}{dy}+\frac{du}{dz}=\frac{3}{x+y+z}$.

(e) Evaluate $\lim_{x\to 1} x^{\frac{1}{1-x}}$.

(f) Verity Cauchy's mean values theorem for the functions f(x) = sin x, g (x) = cos x in $\left[0, \frac{\pi}{2}\right]$.

(g) Find the equation of the normal to the curve $y=(\log x)^2$ at $x=\frac{1}{e}$.

3. Answer any five questions : [4x6=15]

(a) Evaluate $\int_{-1}^{2}\{|x|+[x]\}\,dx$.

(b) Evaluate : $\int_0^a x^2(a^2-x^2)^{\frac{5}{2}}dx$

(c) Evaluate : $\int \frac{dx}{e^{4x}-5}$

(d) Evaluate : $\int x^2 \tan^{-1} x\,dx$.

(e) Find the particular solution of the following differential equation.

$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$ given that $y = \sqrt{3}$ when x=1.

(f) Solve $(x + 2y^3)\frac{dy}{dx} = y$.

(f) Solve $\operatorname{cosec} x \frac{d^2y}{dx^2} = x$.

4. Answer any Five questions : [3x5=15]

(a) Prove that the vectors $2\hat{i} - \hat{j} + \hat{k}$, $\hat{i} - 3\hat{j} - 5\hat{k}$ $3\hat{i} - 4\hat{j} - 4\hat{k}$ are the slides of a right angled triangle.

(b) It $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, then verify that $\vec{a} x \vec{b}$ is perpendicular to both $\vec{a}$ and $\vec{b}$.

(c) If $\vec{p} = \frac{1}{\lambda}(\vec{b} x \vec{c})$, $\vec{q} = \frac{1}{\lambda}(\vec{c} x \vec{a})$ and $\vec{r} = \frac{1}{\lambda}(\vec{a} x \vec{b})$ where $\lambda = [\vec{a}\ \vec{b}\ \vec{c}] \neq 0$, then show that $(\vec{a} + \vec{b} + \vec{c}) . (\vec{p} + \vec{q} + \vec{r}) = 3$.

(d) Find the equation of the plane through the point (2, 1, 0) and passing through the intersections of the planes 3x-2y+z-1=0 and x-2y+3z=1.

(e) Find the coordinates of the point where the perpendicular from the origin to the line joining the points (-9, 4, 5) and (11, 0,-1) meets the line.

(f) Find the value of a for which the plane $x + y + z - a = 0$ will touch the sphere $x^2 + y^2 + z^2 - 2x - 2y - 6 = 0$.

(g) Find the feasible region of the system $2y - x \geq 0, 6y - 3x \leq 21, x \geq 0, y \geq 0$.

(h) Solve the following L.P.P. Graphically :

Maximize : $z = 20x + 30y$

Subject to $3x + 5y \leq 15, x, y \geq 0$.

5. Answer any five questions : [3x5=15]

(a) Solve by Cramer's rule :

$2x - y = 2$

$3x + y = 13$

(b) If the matrix A is such that

$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} A = \begin{bmatrix} -4 & 1 \\ 7 & 7 \end{bmatrix}$, find A.

(c) Find the inverse of the following matrix.

$$\begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$$

(d) If $P(n-1, 3) : P(n+1, 3) = 5 : 12$, find n.

(e) A cricket team consisting of 11 players is to be chosen from 8 batsman and 5 bowlers. In how many ways can the team be chosen so as to include at least 3 bowlers ?

(f) If x^p occurs in the expansion of

$$\left(x^2 + \frac{1}{x}\right)^{2n}$$

prove that its coefficient is

$$\frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{2n+p}{3}\right)!}$$

(g) Five boys and four girls randomly stand in a line. Find the probability that no two girls come together.

(h) If a random variable x has a bionomial distribution $B\left(8, \frac{1}{2}\right)$ then find x for which the outcome is most likely.

6. Answer any one questions : **[$7^1/_2$]**

(a) If $e^{\frac{1}{x}} = \frac{x}{a+bx}$ then show that

$$x^3 \frac{d^2y}{dx^2} = \left(x\frac{dy}{dx} - y \right)^2.$$

(b) A cylindrical open water bank with a circular base is to be made out of 300 sq metres of metal sheet. Find the dimensions so that it can hold maximum water. (Neglect the thickness of sheet.)

GROUP - C

(Marks : 30)

7. Answer any one question : **[$7^1/_2$]**

(a) Evaluate $\int \frac{dx}{\cos x(1+2\sin x)}$

(b) Solve $\frac{dx}{dy} = \frac{3x-7y+7}{3y-7x-3}$

8. Answer any one question : **[$7^1/_2$]**

(a) (i) Prove the following by vector method: An angle inscribed in a semicircle is a right angle.

(ii) Show that

$$\hat{i}\, x(\vec{a}\, x\, \hat{i}) + \hat{j}\, x(\vec{a}\, x\, \vec{j}) + \hat{k}\, x(\vec{a}\, x\, \vec{k}) = 2\vec{a}$$

(b) Show that the line joining the points (0, 2, -4) and (-1, 1, -2) and the line joining the points (-2, 3, 3) and (-3, -2, 1) are coplanar. Find their point of intersection.

(c) Solve the following L.P.P. graphically

Maximize $Z = 4x_1 + 3x_2$

Subject to $x_1 + x_2 \le 50$

$x_1 + 2x_2 \le 80$

$2x_1 + x_2 \ge 20$

$x_1 + x_2 \ge 0$.

9. Answer any one question : **[$1x7^1/_2$]**

(a) Prove that

$$\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a+b+c)$$

$$(b-c)(c-a)(a-b)$$

(b) Show that

$$C_1^2 + 2C_2^2 + 3C_3^2 + ... + nC_n^2 = \frac{(2n-1)!}{\{(n-1)!\}^2}$$

(c) (i) The probability that a student will pass the final examination in both English and Hindi is 0.5 and the probability of passing neither is 0.1. If the probability of passing English examination 0.75, what is the probability of passing the Hindi examination ?

(ii) If $P(A) = 0.4$, $P(B/A) = 0.3$ and $P(B^C / A^C) = 0.2$ then find $P(A/B)$.

***

ANSWERS

2017 (A) - ANSWERS

GROUP - A

1. (a) The highest power of x is $(3x^3+7)^{15}$ is 45.

$$\therefore \frac{d^{45}(3x^2+7)^{15}}{dx^{45}} = \text{Constant}$$

$$\therefore \frac{d^{46}(3x^2+7)^{15}}{dx^{46}} = 0$$

The minimum value of n is 46.

(b) Let $f(x) = \sin^2 x - x$

$\therefore f'(x) = 2\sin x\cos x - 1$

$= \sin 2x - 1 \le 0$ for all $n \in R$.

The function is decreasing in R

$\Rightarrow$ The function is increasing in the interval ϕ.

(c) $\int_0^1 \{x\}dx = \int_0^1 c\ x\,dx$

where c is the fractional part of x.

$= c\int_0^1 x\,dx$

$= c\left[\frac{x^2}{2}\right]_0^1 = c$

$\Rightarrow \int_0^1 \{x\}dx =$ The fractional part of x.

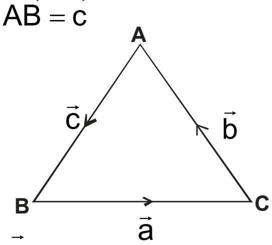
(d) The equation of the family of circles is

$ax^2 + ay^2 + 2gx + 2fy + c = 0$

$\Rightarrow x^2 + y^2 + \frac{2q}{a}x + \frac{2f}{b}y + \frac{c}{a} = 0$

This equation contains 3 arbitrory constants. Thus the differential equation is of order 3.

(e) Given that $\overline{BC} = \vec{b}$, $\overline{AB} = \vec{c}$



$\overline{BC} + \overline{CA} + \overline{AB} = 0$

$\Rightarrow \vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c}$

$\Rightarrow (\vec{a} + \vec{b})\ x\ \vec{c} = -\vec{c}\ x\ \vec{c}$

$\Rightarrow \vec{a}\ x\ \vec{c} + \vec{b}\ x\ \vec{c} = \vec{0}$

(f) Let OA = p, OB = q, OC = r

The coordinates of A, B and C are (p, o, o), (o, q, o) and (o, o, r).

Let the centroid of the triangle ABC be (a, b, c)

$\therefore a = \frac{p+o+o}{3} = \frac{p}{3}$

$b = \frac{o+q+o}{3} = \frac{q}{3}$

$c = \frac{o+o+r}{3} = \frac{r}{3}$.

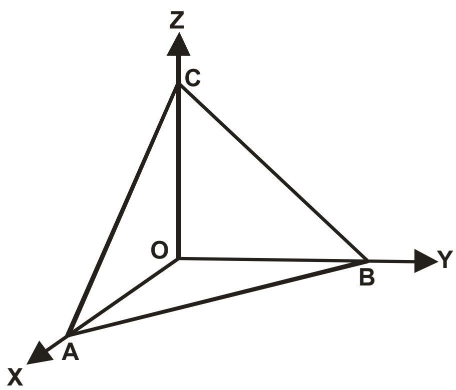
$\Rightarrow p = 3a,\ q = 3b,\ r = 3c$

The equation of the plane is

$\frac{x}{p} + \frac{y}{q} + \frac{z}{r} = 1$

$\Rightarrow \frac{x}{3a} + \frac{y}{3b} + \frac{z}{3c} = 1$

$\Rightarrow \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.



(g) The equation of the line is

$$\frac{x-4}{1}=\frac{y-2}{1}=\frac{z-k}{2} \quad \text{... (1)}$$

The given plane is $2x-4y+z=7$ ... (2)

(4,2,k) is a point on the line (1).

Since the line (1) lies on the plane (2), the point (4, 2, k) also lies on the plane (2).

$\therefore 2.4-4.2+k=7$

$\Rightarrow 8-8+k=7$

$\Rightarrow k=7$

(h) I_n is an identity matrix of order n.

k is any natural number.

$I_n^k = I_n \times I_n \times \ldots k$ times $= I_n$

(i) 5 boys and 5 girls set around a round table. Total number of boys and girls is 10. The number of boys in which they sit around a round table = (10-1)! = 9!

(j) There is a pack of 52 cards. One card is drawn. Let S be the sample sapce.

$|S| = C(52,1) = 52$

Let K and S be the events of getting a King and Spade.

$P(K)=\frac{4}{52},\ P(S)=\frac{13}{52}, P(K\cap S)=\frac{1}{52}$

$P(K\cup S)=P(K)+P(S)-P(K\cap S)$

$=\frac{4}{52}+\frac{13}{52}-\frac{1}{52}=\frac{16}{52}=\frac{4}{13}.$

GROUP - B

2. (a) Let $y=\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ and $z=\sqrt{1-x^2}$

We shall find out $\frac{dy}{dz}$.

Let $x=\cos\theta \Rightarrow \theta=\cos^{-1}x$

$\therefore y=\sec^{-1}\left(\frac{1}{2\cos^2\theta-1}\right)$

$=\sec^{-1}\left(\frac{1}{\cos 2\theta}\right)=\sec^{-1}(\sec 2\theta)$

$=2\theta=2\cos^{-1}x$

$\frac{dy}{dx}=2\frac{d\cos^{-1}x}{dx}=2\left(-\frac{1}{\sqrt{1-x^2}}\right)=-\frac{2}{\sqrt{1-x^2}}$

$z=\sqrt{1-x^2}$

$\frac{dz}{dx}=\frac{d\sqrt{1-x^2}}{dx}=\frac{d(1-x^2)^{\frac{1}{2}}}{dx}$

$=\frac{d(1-x^2)^{\frac{1}{2}}}{d(1-x^2)}.\frac{d(1-x^2)}{dx}=\frac{1}{2}(1-x^2)^{-\frac{1}{2}}.(-2x)$

$=-\frac{x}{\sqrt{1-x^2}}$

$\frac{dy}{dx}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{\frac{-2}{\sqrt{1-x^2}}}{\frac{-x}{\sqrt{1-x^2}}}=\frac{z}{x}.$

(b) $y=x^4e^{2x}=e^{2x}.x^4$

$=uv$ where $u=e^{2x}, v=x^4$

$y_n=D^n(u.v)$

$=u_nv+{}^nc_1u_{n-1}v_1+{}^nc_2u_{n-2}v_2$

$+{}^nc_3u_{n-3}v_3+{}^nc_4u_{n-4}v_4+\ldots$

$v=x^4$

$v_1=4x^3,\ v_2=12x^2, v_3=24x, V_4=24$

$v_5=0=v_6=\ldots$

$u=e^{2x}$

$\therefore u_n=2^ne^{2x}$

$u_{n-1}=2^{n-1}e^{ex}$

$u_{n-2}=2^{n-2}e^{ex}$

$u_{n-3}=2^{n-3}e^{ex}$

$u_{n-4}=2^{n-4}e^{ex}$

From (1), we get

$y_n=2^n e^{2x}.x^4+n2^{n-1}e^{2x}.4x^3$

$+\frac{n(n-1)}{21}2^{n-2}e^{2x}.12x^2$

$+\frac{n(n-1)(n-2)}{3!}2^{n-3}e^{2x}.24x$

$+\frac{n(n-1)(n-2)(n-3)}{4!}2^{n-4}\bullet e^{2x}\bullet 24+0$

$= 2^n x^4 e^{ex} + 4n\,2^{n-1} e^{2x}.x^3$

$+ \frac{12n(n-1)}{2!} 2^{n-2} e^{2x}.x^2$

$+ \frac{24n(n-1)(n-2)}{3!} 2^{n-3} e^{2x}.x$

$+ \frac{n(n-1)(n-2)(n-3)}{4!} 2^{n-4} 2^{2x}.24$

$= 2^n x^4 e^{2x} + 4n\,2^{n-1} e^{2x}.x^3$

$+ 6n(n-1)x^2 2^{n-2} e^{2x}$

$+ 4n(n-1)(n-2)x\,2^{n-3} e^{2x}$

$+ n(n-1)(n-2)(n-3)2^{n-4} 2^{2x}.$

(c) $y = x + \cfrac{1}{x + \cfrac{1}{x + \ldots \text{to} \infty}}$

$\Rightarrow y = x + \frac{1}{y}$

$\Rightarrow y^2 = xy + 1$

Differentialing both sides w.r.t. x, we get

$\frac{dy^2}{dx} = \frac{d(xy)}{dx} + \frac{d1}{dx}$

$\Rightarrow 2y\frac{dy}{dx} = x\frac{dy}{dx} + y.1$

$\Rightarrow 2y\frac{dy}{dx} - x\frac{dy}{dx} = y$

$\Rightarrow (2y - x)\frac{dy}{dx} = y$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2y - x}$

(d) $u = \ln(x^3 + y^3 + z^3 - 3xyz)$

$\frac{du}{dx} = \frac{d\ln(x^3 + y^3 + z^3 - 3xyz)}{d(x^3 + y^3 + z^3 - 3xyz)}.$

$\frac{d(x^3 + y^3 + z^3 - 3xyz)}{dx}$

$= \frac{1}{x^3 + y^3 + z^3 - 3xyz}.(3x^2 - 3yz)$

$= \frac{3(x^2 - yz)}{x^3 + y^3 + z^3 - 3xyz}$

Similarly

$\frac{du}{dy} = \frac{3(y^2 - zx)}{x^3 + y^3 + z^3 - 3xyz},$

$\frac{du}{dy} = \frac{3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz}$

$\therefore \frac{du}{dx} + \frac{du}{dy} + \frac{dy}{dz} =$

$\frac{3(x^2 - yz) + 3(y^2 - zx) + 3(z^2 - xy)}{x^3 + y^3 + z^3 - 3xyz}$

$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x + y + z)(x^2 + y^2 - xy - yz - zx)} = \frac{3}{x + y + z}$

(e) Let $y = \lim_{x \to 1} x^{\frac{1}{1-x}}$ (form is 1^∞)

$= \lim_{x \to 1} e^{\ln\left(x^{\frac{1}{a-x}}\right)} = \lim_{x \to 1} e^{\frac{\ln x}{1-x}}$

$= e^{\lim_{x \to 1}} \left(\frac{\ln x}{1 - x}\right) = (\text{form is } \frac{0}{0})$

$= e^{\lim_{x \to 1}} \frac{\frac{1}{x}}{-1} = e^{-1}.$

(f) $f(x) = \sin x, g(x) = \cos x$

The given interval is $\left[0, \frac{\pi}{2}\right]$

(i) Here f(x) and g(x) are continuous in the closed interval $\left[0, \frac{\pi}{2}\right]$.

(ii) f(x) and g(x) are differentiable in the open interval $\left[0, \frac{\pi}{2}\right]$.

(iii) $g'(x) = -\sin x \neq 0$ in the open interval $\left(0, \frac{\pi}{2}\right)$.

There exists a point c in the open interval $\left(0, \frac{\pi}{2}\right)$ such that .

$\frac{f\left(\frac{\pi}{2}\right) - f(0)}{g\left(\frac{\pi}{2}\right) - g(0)} = \frac{f'(c)}{g'(c)}$

$$\Rightarrow \frac{\sin\frac{\pi}{2}-\sin 0}{\cos\frac{\pi}{2}-\cos 0}=\frac{\cos c}{-\sin c}$$

$$\Rightarrow \frac{1}{-1}=-\cot c$$

$$\Rightarrow \cot c = 1 \Rightarrow c=\frac{\pi}{4}$$

which lies in $\left(0,\frac{\pi}{2}\right)$

So Cauchy's theorem is verified.

(g) The equation of the curve is

$y=(\log x)^2$

When $x=\frac{1}{e}$, then $y=\left(\log\frac{1}{e}\right)^2$

$=\left(\log e^{-1}\right)^2=\left(-\log e\right)^2=1$

The point is $\left(\frac{1}{e},1\right)$

$$\frac{dy}{dx}=\frac{d(\log x)^2}{dx}=2\log x-\frac{1}{x}=\frac{2\log x}{x}$$

At the point

$$\left(\frac{1}{e},1\right),\frac{dy}{dx}=\frac{2\log\frac{1}{e}}{\frac{1}{e}}=2e\log e^{-1}=-2e$$

Let the slope of the normal be m.

$$\therefore m x-2e=-1\Rightarrow \frac{1}{2e}$$

Equation of the normal at $\left(\frac{1}{2e},1\right)$ is

$$y-1=m\left(x-\frac{1}{e}\right)$$

$$\Rightarrow y-1=\frac{1}{2e}\left(x-\frac{1}{e}\right)$$

3. (a) $\int_{-1}^{2}\{|x|+|x|\}\,dx$

$$=\int_{-1}^{2}|x|\,dx+\int_{-1}^{2}[x]\,dx$$

$$=\int_{-1}^{0}|x|\,dx+\int_{0}^{2}|x|\,dx+$$

$$\int_{-1}^{0}[x]\,dx+\int_{0}^{1}[x]\,dx+\int_{1}^{2}[x]dx$$

$$=-\int_{-1}^{0}x\,dx+\int_{0}^{2}x\,dx+$$

$$\int_{-1}^{0}-1dx+\int_{0}^{1}0.dx+\int_{1}^{2}1.dx$$

$$=-\left[\frac{x^2}{2}\right]_{-1}^{0}+\left[\frac{x^2}{2}\right]_{0}^{2}-[x]_{-1}^{0}+[x]_{1}^{2}$$

$$=-\left(-\frac{1}{2}\right)+2-1+1$$

$$=\frac{1}{2}+2-1+1=\frac{5}{2}$$

(b) $\int_0^a x^2(a^2-x^2)^{\frac{5}{2}}dx$

Let $x=a\sin\theta$

$\therefore dx=a\cos\theta\, d\theta$

When $x=0,\ a\sin\theta=a\Rightarrow\theta=0$

When $x=a,\ a\sin\theta=a\Rightarrow\theta=\frac{\pi}{2}$

$$I=\int_0^{\pi/2}a^2\sin^2\theta(a^2-a^2\sin^2\theta)^{5/2}.a\cos\theta\, d\theta$$

$$=a^8\int_0^{\pi/2}\sin^2\theta\cos^6\theta\ d\theta$$

$$=a^8\frac{\sqrt{\left(\frac{2+1}{2}\right)}\sqrt{\left(\frac{6+1}{2}\right)}}{2\sqrt{\left(\frac{2+6+2}{2}\right)}}=\frac{5a^8\pi}{256}.$$

(c) $\int\frac{dx}{e^{4x}-5}=\int\frac{e^{-4x}dx}{1-5e^{-4x}}$

Let $1-5e^{-4x}=t$

$-5.e^{-4x}.(-4dx)=dt$

$$\Rightarrow e^{-4x}dx=\frac{1}{20}dt$$

$$I=\int\frac{1}{20}\frac{dt}{t}=\frac{1}{20}\ln t+c$$

$$=\frac{1}{20}\ln(1-5e^{-4x})+c$$

(d) $\int x^2 \tan^{-1} x \, dx$

$= \int x^2 \tan^{-1} x \,. x^2 \, dx$

$= \tan^{-1} x . \int x^2 dx - \int \left[\frac{d \tan^{-1} x}{dx} . \int x^2 \, dx \right]$

$= \frac{x^3}{3} \tan^{-1} x - \frac{1}{3} \int \frac{x^3}{1+x^2} dx$

$= \frac{1}{3} x^3 \tan^{-1} x - \frac{1}{3} . \frac{x^2}{2} + \frac{1}{6} \ln(1+x2) + c$.

(e) The given equation is $\frac{dy}{dx} = \frac{1+y^2}{1+x^2}$

$\Rightarrow \frac{dy}{1+y^2} = \frac{dx}{1+x^2}$

Integrating both sides, we get

$\int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$

$\Rightarrow \tan^{-1} y = \tan^{-1} x + C$ ... (1)

When $x = 1, y = \sqrt{3}$

From (1) we get $C = \frac{\pi}{12}$.

The required particular solution is

$\tan^{-1} y = \tan^{-1} x + \frac{\pi}{12}$.

(f) The given differential equation is

$\left(x + 2y^3\right) \frac{dy}{dx} = y$

$\Rightarrow y \frac{dx}{dy} = x + 2y^3$

$\Rightarrow \frac{dx}{dy} - \frac{1}{y} x = 2y^2$ ... (1)

Integrating factor

$= \bar{e}^{\int \frac{1}{y} dy} = \bar{e}^{\int \frac{1}{y} dy} = e^{\ln \frac{1}{y}} = \frac{1}{y}$

Multiplying both sides of (1) by $\frac{1}{y}$, we get

$\frac{1}{y} . \frac{dx}{dy} - \frac{1}{y^2} x = 2y$

$\Rightarrow \frac{d}{dy} \left(\frac{x}{y} \right) = 2y.$

Integrating both sides, we get

$\frac{x}{y} = y^2 + C$.

(g) $\text{Co sec } x \frac{d^2y}{dx^2} = x$

$\Rightarrow \frac{d^2y}{dx^2} = x \sin x$

Integrating both sides, we get

$\frac{dy}{dx} = \int x \sin x \, dx$

$= x(-\cos x) - \int 1(-\cos x) dx$

$= -x \cos x + \sin x + C$

$\Rightarrow dy = (-x \cos x + \sin x + C) dx.$

Integrating both sides, we get

$y = -\int x \cos x \, dx + \int \sin x \, dx + C \int dx$

$= -x \sin x - 2 \cos x + cx + d$.

4. (a) Let $\vec{a} = 2\hat{i} - \hat{j} + \hat{k}$

$\vec{b} = \hat{i} - 3\hat{j} - 5\hat{k}$

$\vec{c} = 3\hat{i} - 4\hat{j} - 4\hat{k}$

$\vec{a} + \vec{b} = 2\hat{i} - \hat{j} + \hat{k} + \hat{i} - 3\hat{j} - 5\hat{k}$

$= 3\hat{i} - 4\hat{j} - 4\hat{k} = \vec{c}$

Thus $\vec{a}$, $\vec{b}$, $\vec{c}$ from the sides of a triangle

$\vec{a} \bullet \vec{b} = (2\hat{i} - \hat{j} + \hat{k}) \bullet (\hat{i} - 3\hat{j} - 5\hat{k})$

= 2 + 3 - 5 = 0

$\Rightarrow \vec{a}$ is perpendicular to $\vec{b}$.

So $\vec{a}, \vec{b}, \vec{c}$ form the sides of a right angled triangle.

(b) $\vec{a} = 3\hat{i} + \hat{j} + 2\hat{k}$

$\vec{b} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 3 & 1 & 2 \\ 2 & -3 & 4 \end{vmatrix}$$

$= \hat{i}(4+6) - \hat{j}(12-4) + \hat{k}(-9-2)$

$= 10\hat{i} - 8\hat{j} - 11\hat{k}$

$(\vec{a} \times \vec{b}) \bullet \vec{a} = (10\hat{i} - 8\hat{j} - 11\hat{k}) \bullet (3\hat{i} + \hat{j} + 2\hat{k})$

= 0

$\Rightarrow \vec{a} \times \vec{b}$ is perpendicular to a.

(c) Given that

$\vec{p} = \frac{1}{\lambda}(\vec{b} \times \vec{c}),\ \vec{q} = \frac{1}{\lambda}(\vec{c} \times \vec{a})$

$\vec{r} = \frac{1}{\lambda}(\vec{a} \times \vec{b})$

$(\vec{a} + \vec{b} + \vec{c}) \bullet (\vec{p} + \vec{q} + \vec{r})$

$= \vec{a} \bullet \vec{p} + \vec{a} \bullet \vec{q} + \vec{a} \bullet \vec{r} + \vec{b} \bullet \vec{p} \bullet \vec{b} \bullet \vec{q}$

$+ \vec{c} \bullet \vec{p} + \vec{c} \bullet \vec{q} + \vec{c} \bullet \vec{r}$

$= \vec{a} \bullet \frac{1}{\lambda}(\vec{b} \times \vec{c}) + \vec{a} \bullet \frac{1}{\lambda}(\vec{c} \times \vec{a}) + \vec{a} \bullet \frac{1}{\lambda}(\vec{a} \times \vec{b})$

$+ \vec{b} \bullet \frac{1}{\lambda}(\vec{b} \times \vec{c}) + \vec{b} \bullet \frac{1}{\lambda}(\vec{c} \times \vec{a}) + \vec{b} \bullet \frac{1}{\lambda}(\vec{a} \times \vec{b})$

$+ \vec{c} \bullet \frac{1}{\lambda}(\vec{b} \times \vec{c}) + \vec{c} \bullet \frac{1}{\lambda}(\vec{c} \times \vec{a}) + \vec{c} \bullet \frac{1}{\lambda}(\vec{a} \times \vec{b})$

$= \frac{1}{\lambda}\left[\vec{a}\vec{b}\vec{c}\right] + 0 + 0 + 0$

$+ \left[\vec{b}\vec{c}\vec{a}\right] + 0 + 0 + 0 + \left[\vec{c}\vec{a}\vec{b}\right]$

$= \frac{1}{\lambda}3\left[\vec{a}\vec{b}\vec{c}\right] = 3.$

(d) Two given planes are

$3x - 2y + z - 1 = 0$ ... (1)

$x - 2y + 3z - 1 = 0$ ... (2)

Equation of the plane passing through the intersection of (1) and (2) is

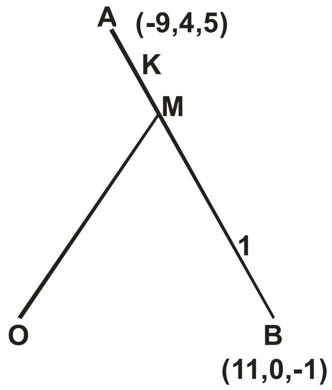
$(3x - 2y - z - 1) + k(x - 2y + 3z - 1) = 0$

Since it is passing through (2,1,0), we have (3.2 - 2.1 - 0 -1) + k (2 - 2. 1 + 3.0 - 1)=0

$= 3x - 4y + 5z - 2 = 0$

(e) Let A and B be two points whose coordinates are (-9, 4, 5) and (1, 0, -1) respectively.

Let 0 be the origin.



Let OM be the perpendicular form 0 to the line AB. We shall find the coordinates of M.

Let M divides of M are

Let M divides AB in the ratio k : 1.

The coordinates of M are

$$\left(\frac{11k-9}{k+1}, \frac{k.0+4}{k+1}, \frac{-k+5}{k+1}\right) =$$

$$\left(\frac{11k-9}{k+1}, \frac{4}{k+1}, \frac{-k+5}{k+1}\right) \quad ...(1)$$

The d.r.s of OM are

$$\left\langle \frac{11k-9}{k+1} - 0, \frac{4}{k+1} - 0, \frac{-k+5}{k+1} - 0 \right\rangle$$

$$= \left\langle \frac{11k-9}{k+1}, \frac{4}{k+1}, \frac{-k+5}{k+1} \right\rangle$$

The d.r s of AB are <11-(-9), 0-4, -1-5>

$\langle 11-(-9), 0-4, -1-5 \rangle = \langle 20, -4, -6 \rangle$.

Since OM is perpendicular to AB, we have

$$20\left(\frac{11k-9}{k+1}\right) + (-4) \bullet \frac{4}{k+1} + (-6)\left(\frac{-x+5}{x+1}\right) = 0$$

$\Rightarrow k = 1$.

From (1), the coordinates of M are

$$\left(\frac{11.1-9}{1+1}, \frac{4}{1+1}, \frac{-1+5}{1+1}\right) = (1,2,2).$$

(f) The given plane is x+y+z-1=0 ...(1)

The given sphere is

$x^2 + y^2 + z^2 - 2x - 2y - 6 = 0$

$\Rightarrow (x-1)^2 + (y-1)^2 + (2-1)^2 = 3^2$ ... (2)

Enter of the sphere is (1, 1, 1) and radius =3.

If the plane (1) touches the sphere (2) then the perpendicular from the centre of the plane = radius.

$\Rightarrow \frac{1+1+1-1}{\sqrt{1^2+1^2+1^2}} = 3$

$\Rightarrow \frac{3-a}{\sqrt{3}} \Rightarrow a = 3 \pm 3\sqrt{3}$.

(g) Given in equation are

$2y - x \geq 0$ ...(1)

$6y - 3x \leq 21$ ...(2)

$x \geq 0,\ y \geq 0$ ...(3)

Converting the inequations (1) and (2) to equations, we get

$2y - x = 0$ ...(4)

$6y - 3x = 21$ ...(5)

From equation (4), we get

x	0	2
y	0	1

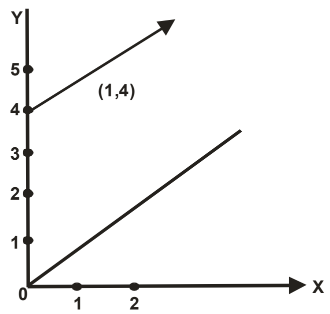
The line (4) passes through (0,0) and (2,1).

From equation (5) we get

x	1	3
y	4	5

The line (5) passes through (1,4) and (3,5).

The shape of the feasible region is given below. The feasible region is between two parallel lines.



(h) The given L.P.P. is

Maximize $z = 20x + 30y$ ... (1)

Subject to $3x + 5y \leq 15$ ... (2)

$x, y \geq 0$ ... (3)

Transforming the inequation (2) to equation, we get $3x + 5y = 15$ ...(4)

From (4) we see that

x	0	5
y	3	0

The line (4) passes through (0, 3) & (5, 0) putting (0, 0) is (2), we get

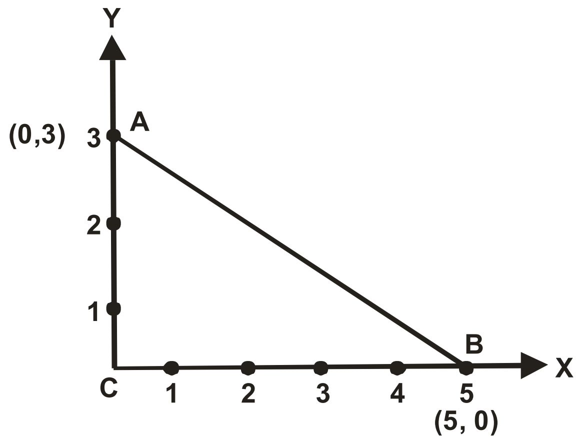
$3.0 + 5.0 \leq 15$

$\Rightarrow 0 \leq 15$ which is true.

So half plane is towards the origin.

The feasible region is in the first quadrant.

The feasible region is shown in the following figure. The value of Z at the corner points are given in the following table.



Point	x	y	$z = 20x + 30y$
0	0	0	$z = 0$
A	5	0	$z = 20x5 + 30.0 = 100$
B	0	3	$z = 20x0 + 30x3 = 90$

The maximum value of Z is 100 and it is obtained when x = 5, y = 0.

5. (a) The given equations are

$2x - y = 2$

$3x + y = 13$

Cramer's Rule is

$$\frac{x}{Dx} = \frac{y}{Dy} = \frac{1}{D} \quad \text{... (1)}$$

Where

$$Dx = \begin{vmatrix} 2 & -1 \\ 13 & 1 \end{vmatrix} = 2 - (-13) = 15$$

$$Dy = \begin{vmatrix} 2 & 2 \\ 3 & 13 \end{vmatrix} = 26 - 6 = 20$$

$$D = \begin{vmatrix} 2 & -1 \\ 3 & 1 \end{vmatrix} = 2 - (-3) = 5$$

From (1), we get $\frac{x}{15} = \frac{y}{20} = \frac{1}{5}$

$$\Rightarrow x = \frac{15}{5} = 3$$

$$y = \frac{20}{5} = 4.$$

(b) The given matrix equation is

$$\begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} A = \begin{bmatrix} -4 & 1 \\ 7 & 7 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} -4 & 1 \\ 7 & 7 \end{bmatrix} \quad \text{...(1)}$$

Let $B = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix}$

B_{11} = Cofactor of $1 = (-1)^{1+1}.3 = 3$

B_{12} = Cofactor of $-1 = (-1)^{1+2}\, 2 = -2$

B_{21} = Cofactor of $2 = (-1)^{2+1}(-1) = -(-1) = 1$

B_{22} = Cofactor of $3 = (-1)^{2+2}.1 = 1.$

Matrix of cofactors $= \begin{bmatrix} 3 & -2 \\ 1 & 1 \end{bmatrix}$

Adjoint of $B = \begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$

$$|B| = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 - (-2) = 5$$

$$B^{-1} = \frac{1}{|B|} \text{ adj } B = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}$$

Form (1), we get

$$A = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 7 & 7 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} -12+7 & 3+7 \\ 8+7 & -2+7 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} -5 & 10 \\ 15 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}.$$

(c) Let $A = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix}$

$$|A| = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 0 \end{bmatrix} = 0 - 0 + 2(0-4) = -8$$

A_{11} = Cofactor of $0 = (-1)^{1+1}\begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix} = 0$

$A_{12} = 0$, $A_{13} = -4$, $A_{21} = 0$, $A_{22} = -4$,

$A_{23} = 0$, $A_{31} = -4$, $A_{32} = 0$, $A_{35} = 0$

Matrix of cofactors $= \begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ -4 & 0 & 0 \end{bmatrix}$

Adj A $= \begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ -4 & 0 & 0 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{-8}\begin{bmatrix} 0 & 0 & -4 \\ 0 & -4 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

(d) $p(n-1, 3) : p(n+1, 3) = 5:12$

$$\Rightarrow \frac{p(n-1,3)}{p(n+1,3)} = \frac{5}{12}$$

$$\Rightarrow \frac{(n-1)(n-2)(n-3)}{(n+1)n(n-1)} = \frac{5}{12}$$

$$\Rightarrow \frac{(n-2)(n-3)}{(n+1)n} = \frac{5}{12}$$

$\Rightarrow 7n^2 - 65n + 72 = 0$

$\Rightarrow (n-8)(7n-9) = 0$

$\because n \neq \frac{9}{7}, n = 8$.

(e) There are 8 bats men and 5 bowlers. 11 players are to be chosen so as to include at least 3 bowers.

1st type - 8 batsmen and 3 bowlers

2nd type - 7 batsmen and 4 bowlers

3rd type - 6 batsmen and 5 bowlers.

Total numbers of selection

$= c(8,8) \times c(5,3) + c(8,7) \times c(5,4) + c(8,6) \times c(5,5)$

= 10 + 40 + 28 = 78.

(f) The given expression $= \left(x^2 + \frac{1}{x}\right)^{2n}$

Let x^p occurs in $(4 + 1)^{th}$ term.

$(r + 1)^{th}$ term = $c(2n, r)(x^2)^{2n-4} . \left(\frac{1}{x}\right)^r$

$= c(2n, r)\ x^{n+3r}$

Since this term contains x^p,

$\therefore 4n - 3r = p$

$\Rightarrow 3r = 4n - p$

$\Rightarrow r = \frac{4n-p}{3}$.

The coefficient of

$x^p = c(2n, r)$

$$= \frac{(2n)!}{r!(2n-r)!}$$

$$= \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(2n - \frac{4n-p}{3}\right)!}$$

$$= \frac{(2n)!}{\left(\frac{4n-p}{3}\right)!\left(\frac{4n+p}{3}\right)!}$$

(g) There are 5 boys and 4 girls. They should stand on a line such that no two girls come together. 4 girls should stand in between boys. There are 4 places in between boys and 2 places in each extreme.

6 places are occupied by 4 girls in p(6,4) way. Again 5 boys are arranged among themselves in p(5, 5) ways.

Total number of arrangement.

$= p(6,4) \times p(5,5)$

$$= \frac{6!}{(6-4)!} \times 5!$$

= 6.5.4.3.5.4.3.2.1.

Total number of boys and girls = 5+4 = 9.

Sample space = Total number of arrangement

= p(9,9)

= 9.8.7.6.5.4.3.2.1.

Required probability

$$= \frac{6.5.4.3.\ 5.4.3.2.1}{9.8.7.6.5.4.3.2.1} = \frac{5}{42}$$

(h) Let x be the random variable whose binomial distribution is $B\left(6, \frac{1}{2}\right)$.

$n = 6,\ p = \frac{1}{2}$

$\therefore q = 1 - p = 1 - \frac{1}{2} = \frac{1}{2}$.

We know $p(x = r) = 6_{c_r} p^r q^{6-r}$

$$= 6_{c_r} . \left(\frac{1}{2}\right)^r \left(\frac{1}{2}\right)^{6-r}.$$

$$p(x=0)=6_{c_0}\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^{6-0}=1.\left(\frac{1}{2}\right)^6=\frac{1}{64}$$

$$p(x=1)=6_{c_1}\left(\frac{1}{2}\right)^1\left(\frac{1}{2}\right)^{6-1}=6.\frac{1}{2}.\left(\frac{1}{2}\right)^5=\frac{6}{64}$$

$$p(x=2)=6_{c_2}\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^{6-2}=\frac{6.5}{1.2}.\frac{1}{4}.\frac{1}{8}=\frac{15}{64}$$

$$p(x=3)=6_{c_3}\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^{6-3}$$

$$=\frac{6.5.4}{3.2.1}.\frac{1}{8}.\frac{1}{8}=\frac{20}{64}$$

$$p(x=4)=6_{c4}\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^{6-4}$$

$$=\frac{6.5.4.3}{1.2.3.4}.\frac{1}{16}.\frac{1}{4}=\frac{15}{64}$$

$$p(x=5)=6_{c_5}\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^{6-5}$$

$$=\frac{6.5.4.3.2}{1.2.3.4.5}.\frac{1}{32}.\frac{1}{2}=\frac{6}{64}$$

$$p(x=6)=6_{c_6}\left(\frac{1}{2}\right)^6.\left(\frac{1}{2}\right)^0=\frac{1}{64}.$$

Here we see that $p(x=3)=\frac{20}{64}$ is maximum of all the above values.

This means x=3 is the most likely outcome.

GROUP - C

6. (a) $e^{\frac{y}{x}}=\frac{x}{a+bx}$

$$\Rightarrow y=x[\ln x-\ln(a+bx)]$$

$$\frac{dy}{dx}=\ln x-\ln(a+bx)+x\left(\frac{1}{x}-\frac{b}{a+bx}\right)$$

$$=\ln\frac{x}{a+bx}+1-\frac{bx}{a+bx} \qquad ...(1)$$

$$=\frac{y}{x}+1-\frac{bx}{a+bx}$$

$$\Rightarrow x\frac{dy}{dx}=y+x-\frac{bx^2}{a+bx}$$

$$\Rightarrow x\frac{dy}{dx}-y=x\left(1-\frac{bx}{a+bx}\right)$$

$$=x\left(\frac{a+bx-bx}{a+bx}\right)=\frac{ax}{a+bx}$$

$$\therefore\left(x\frac{dy}{dx}-y\right)^2=\frac{a^2x^2}{(1+bx)^2} \qquad ...(2)$$

Again from (2), we get

$$\frac{dy}{dx}=\ln x-\ln(a+bx)+\frac{a}{a+bx}$$

$$\therefore\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{1}{x}-\frac{b}{a+bx}-\frac{ab}{(a+bx)^2}$$

$$\Rightarrow\frac{d^2y}{dx^2}=\frac{a+bx-bx}{x(a+bx)}-\frac{ab}{(a+bx)^2}$$

$$=\frac{a}{x(a+bx)}-\frac{ab}{(a+bx)^2}$$

$$=\frac{a}{a+bx}\left(\frac{1}{x}-\frac{b}{a+bx}\right)$$

$$=\frac{a}{a+bx}\left(\frac{a+bx-bx}{x(a+bx)}\right)$$

$$=\frac{a^2}{x(a+bx)^2}$$

$$x^3\frac{d^2y}{dx^2}=\frac{a^2x^3}{x(a+bx)^2}=\frac{a^2x^2}{(a+bx)^2} \qquad ...(3)$$

From (2) and (3), we get

$$x^3\frac{d^2y}{dx^2}=\left(x\frac{dy}{dx}-y\right)^2.$$

(b) Let 4 be the radius of the base and h be the height of the cylindrical open tank.

Surface area $= \pi r^2 + 2\pi rh$ ...(1)

According to the question,

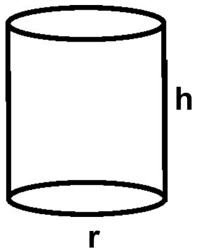
$\pi r^2 + 2\pi rh = 30$

$\Rightarrow 2\pi rh = 30 - \pi r^2$

$\Rightarrow h = \dfrac{30 - \pi r^2}{2\pi r}$

Let v be the volume of cylindrical open tank.

$V = \pi r^2 h = \pi r^2 . \left(\dfrac{30 - \pi r^2}{2\pi r}\right)$



$= \dfrac{1}{2}(30r - \pi r^3)$

$\dfrac{dv}{dr} = \dfrac{1}{2}(30 - 3\pi r^2)$

$\dfrac{d^2v}{dr^2} = \dfrac{1}{2}(0 - 6\pi r) = -3\pi r$.

For maximum or minimum value of V, we have

$\dfrac{dv}{dr} = 0$

$\Rightarrow 30 - 3\pi r^2 = 0$

$\Rightarrow 3\pi r^2 = 30$

$\Rightarrow r^2 = \dfrac{30}{3\pi} = \dfrac{10}{\pi}$

$\Rightarrow r = \sqrt{\dfrac{10}{\pi}}$.

When $r = \sqrt{\dfrac{10}{\pi}}, \dfrac{d^2v}{dr^2} = -3\pi\sqrt{\dfrac{10}{\pi}}$

which is -ve.

V is maximum, when $r = \sqrt{\dfrac{10}{\pi}}$,

$\therefore h = \dfrac{30 - \pi . \dfrac{10}{\pi}}{2\pi\sqrt{\dfrac{10}{\pi}}} = \sqrt{\dfrac{10}{\pi}}$

Thus the volume π maximum when

radius = height = $\sqrt{\dfrac{10}{\pi}}$.

7. (a) Let

$I = \int \dfrac{dx}{\cos x(1 + 2\sin x)}$

$= \int \dfrac{\cos\ x\ dx}{\cos^2 x(1 + 2\sin x)}$

$= \int \dfrac{\cos\ x\ dx}{(1 - \sin^2 x)(1 + 2\sin x)}$

$= \int \dfrac{\cos\ x\ dx}{(1 + \sin x)(1 - \sin x)(1 + 2\sin x)}$

$= \int \dfrac{1}{(1 + t)(1 - t)(1 + 2t)}\ dt$

$[\sin x = t, \cos x\, dx = dt]$

Let

$\dfrac{1}{(1x + t)(1 - t)(1 + 2t)} = \dfrac{A}{1 + t} + \dfrac{B}{1 - t} + \dfrac{C}{1 + 2t}$ ... (1)

$\Rightarrow 1 = A(1 - t)(1 + 2t) +$

$B(1 + t)(1 + 2t) + C(1 + t)(1 - t)$

$\therefore A = \dfrac{1}{6}, B = -\dfrac{1}{2}, C = \dfrac{4}{3}$

From (1), we have

$\dfrac{1}{(1 + t)(1 - t)(1 + 2t)} =$

$\dfrac{1}{6} . \dfrac{1}{1 + t} - \dfrac{1}{2} . \dfrac{1}{1 - t} + \dfrac{4}{3} . \dfrac{1}{1 + 2t}$

$\int \dfrac{1}{(1 + t)(1 - t)(1 + 2t)} dt =$

$\dfrac{1}{6}\int \dfrac{1}{1 + t} dt - \dfrac{1}{2}\int \dfrac{1}{1 - t} dt + \dfrac{4}{3}\int \dfrac{1}{1 + 2t} dt$

$= \dfrac{1}{6}\ \ln(1 + t) + \dfrac{1}{2}\ \ln(1 - t) + \dfrac{2}{3}\ln(1 + 2t) + C$

$= \dfrac{1}{6}\ \ln(1 + \sin x) + \dfrac{1}{2}\ \ln(1 - \sin x)$

$+ \dfrac{2}{3}\ln(1 + 2\sin x) + C$

(b) $\frac{dx}{dy}=\frac{3x-7y+7}{3y-7x-3}$

$\Rightarrow \frac{dy}{dx}=\frac{3y-7x-3}{3x-7y+7}$ ... (1)

Let $x = X + h,\ y = Y + k$

$\therefore \frac{dy}{dx}=\frac{dy}{dx}$

Equation (1) becomes

$\frac{dY}{dX}=\frac{(3Y-7X)+(3k-7y-3)}{(3X-7Y)+(3h-7k+7)}$ ...(2)

Let us choose h, k such that

3k - 7h - 3 = 0

3h - 7k + 7 = 0

Solving, h=0, k=1

$\therefore x = X,\ Y = y+1$

$\Rightarrow X = x,\ Y = y-1$

Equation (2) becomes,

$\frac{dY}{dx}=\frac{3Y-7X}{3X-7Y}$ ... (3)

Let $Y = VX$

$\therefore \frac{dY}{dX}=V+X\frac{dv}{dx}$

Equation (3) becomes

$V+x\frac{dV}{dX}=\frac{3VX-7X}{3X-7VX}=\frac{3V-7}{3-7V}$

$\Rightarrow X\frac{dV}{dX}=\frac{3V-7}{3-7V}-V$

$=\frac{3V-7-V(3-7V)}{3-7V}$

$=\frac{7V^2-7}{3-7V}=\frac{7(V^2-1)}{3-7V}$

$\Rightarrow \frac{3-7V}{V^2-1}dv=7.\frac{dX}{X}$

Integrating both sides, we get

$\int\frac{3-7V}{V^2-1}dV=7\int\frac{dX}{X}$

$\Rightarrow 3\int\frac{1}{V^2-1}dv-\frac{7}{2}\int\frac{2V}{V^2-1}dV=7X+C$

$\Rightarrow 3.\frac{1}{2}\ln\frac{V-1}{V+1}-\frac{7}{2}\ln(V^2-1)=7X+C$

$\Rightarrow \frac{3}{2}\ln\left(\frac{\frac{Y}{X}-1}{\frac{Y}{X}+1}\right)-\frac{7}{2}\ln\left(\frac{Y^2}{X^2}-1\right)=7X+C$

$\Rightarrow \frac{3}{2}\ln\left(\frac{Y-X}{Y+X}\right)-\frac{7}{2}\ln\left(\frac{Y^2-X^2}{X^2}\right)=7X+C$

$\Rightarrow \frac{3}{2}\ln\left(\frac{Y-X}{Y+X}\right)-7\ln\left(Y^2-X^2\right)+7X=7X+C$

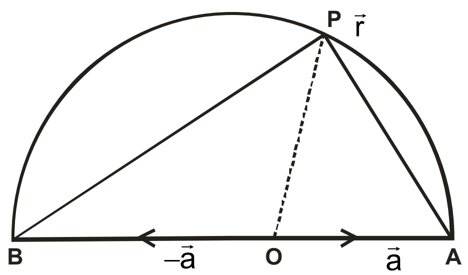
$\Rightarrow \frac{3}{2}\ln\left(\frac{Y-X}{Y+X}\right)-7\ln\left(Y^2-X^2\right)=C$

Where X = x, Y = y - 1.

8.(a)(i) Let 0 be the centre and rbe the radius.

Let $\overrightarrow{OA}=\vec{a}\ \therefore \overrightarrow{OB}=-\vec{a}$

Let P be any point on the semicircle.



$\therefore \overrightarrow{OP}=\vec{r}$

$\overrightarrow{AP}=$ P. V. of P - P. V. of A $=\vec{r}-\vec{a}$

$\overrightarrow{BP}=$ P.V. of P - P.V. of

$B=\vec{r}-(-\vec{a})=\vec{r}+\vec{a}$

$\overrightarrow{AP}\ .\ \overrightarrow{BP}=(\vec{r}-\vec{a}).(\vec{r}+\vec{a})$

$= \vec{r}^2 - \vec{a}^2$

$= |\vec{r}|^2 - |\vec{a}|^2 = 0$

$\Rightarrow$ AP is perpendicular to BP.

$\Rightarrow$ The angle in a semicircle is a right angle.

(ii) We know $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c}$

Let $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$.

L.H.S. = $\hat{i} \times (\vec{a} \times \hat{i}) + \hat{j} \times (\vec{a} \times \hat{j}) + \hat{k} \times (\vec{a} \times \hat{k})$

$= (\hat{i}.\hat{i})\vec{a} - (\hat{i}.\vec{a})\hat{i} + (\hat{j}.\hat{j})\vec{a} - (\hat{j}.\vec{a})\hat{j}$

$+ (\hat{k}.\hat{k})\vec{a} - (\hat{k}.\vec{k})\vec{a} - (\hat{k}.\vec{a})\vec{k}$

$= \vec{a} - (\hat{i}.\vec{a})\hat{i} + \vec{a} - (\hat{j}.\vec{a})\hat{j} + \vec{a} - (\hat{k}.\vec{a})\hat{k}$

$= 3\vec{a} - \left[(\hat{i}.\vec{a})\hat{i} + (\hat{j}.\hat{a})\hat{j} + \vec{a} - (\hat{k}.\vec{a})\hat{k}\right]$

$= 3\vec{a} - [\{\hat{i}(x\hat{i} + y\hat{j} + 2\hat{k})\}\hat{i}$

$+ \{\hat{j}.(x\hat{i} + y\hat{j} + z\hat{k})\}\hat{j}$

$+ \{\hat{k}.(x\hat{i} + y\hat{j} + z\hat{k})\}\hat{k}]$

$= 3\vec{a} - [x\hat{i} + y\hat{j} + z\hat{k}]$

$= 3\vec{a} - \vec{a} = 2\vec{a}$.

(b) Equation of the line passing through (0, 2, -4) and (-1, 1, -2) is

$$\frac{x-0}{-1-0} = \frac{y-2}{1-2} = \frac{z+4}{-2+4}$$

$$\Rightarrow \frac{x}{-1} = \frac{y-2}{-1} = \frac{z+4}{2}$$

$$\Rightarrow \frac{x-0}{-1} = \frac{y-2}{-1} = \frac{z-(-4)}{2} = r_1 \text{ (say) ... (1)}$$

Equation of the line passing through the points (-2, 3, 3) and (-3, -2, 1) is

$$\frac{x+2}{-3+2} = \frac{y-3}{-2-3} = \frac{z-3}{1-3}$$

$$\Rightarrow \frac{x+2}{-1} = \frac{y-3}{-5} = \frac{z-3}{-2}$$

$$\Rightarrow \frac{x-(-2)}{-1} = \frac{y-3}{5} = \frac{z-3}{2} = r_2 \text{ (say)...(2)}$$

The lines (1) and (2) are coplanar if

$$\begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

$$\text{Now} \begin{vmatrix} x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

$$= \begin{vmatrix} -2-0 & 3-2 & 3-(4) \\ -1 & -1 & 2 \\ 1 & 5 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 & 7 \\ -1 & -1 & 2 \\ 1 & 5 & 2 \end{vmatrix}$$

= - 2 (- 2 - 10) - 1 (- 2 - 2) + 7 (-5 + 1)

= - 2 (- 12) - (-4) + 7 (-4)

= 24 + 4 - 28 = 0

So the two lines are coplanar.

Any point on the line (1) is

$(-r_1, -r_1 + 2, 2r_1 - 4)$ ... (3)

Any point on the line (2) is

$(4_2 - 2, 5r_2 + 3, 2r_2 + 3)$ ... (4)

At the point of intersection

$-r_1 = r_2 - 2,$

$-r_1 + 2 = 5r_2 + 3$

$2r_1 - 4 = 2r_2 + 3$

$\Rightarrow r_1 + r_2 - 2 = 0$ ...(5)

$r_1 + 5r_2 + 1 = 0$ ...(6)

$2r_1 2r_2 - 7 = 0$ ...(7)

Solving (5) and (6), we get

$r_1 = \frac{11}{4}, r_2 = -\frac{3}{4}$.

We see that $r_1 = \frac{11}{4}, r_2 = -\frac{3}{4}$ satisfy the equation (7).

The point of intersection is obtained by putting $r_1 = \frac{11}{4}$ in (3) or $r_2 = -\frac{3}{4}$ in (4)

The point of intersection is $\left(-\frac{11}{4}, -\frac{3}{4}, \frac{3}{2}\right)$.

(c) The given L.P.P. is

Maximize $z = 4x_1 + 3x_2$ ... (1)

$x_1 + x_2 \le 50$ ... (2)

$x_1 + 2x_2 \le 80$ ... (3)

$2x_1 + x_2 \ge 20$ ... (4)

$x_1 + x_2 \ge 0$ .. (5)

Transforming the inequations to equations, we get

$x_1 + x_2 = 50$ ... (6)

$x_1 + 2x_2 = 80$ ... (7)

$2x_1 + x_2 = 20$ ... (8)

From equation (6), we see that

x_1	0	50
x_2	50	0

The line (6) passes through two points (50, 0) and (0, 50).

Putting (0, 0) is (2), we get

$0 + 0 \le 50$

$\Rightarrow 0 \le 50$ which is true.

The half plane is towards the origin.

Similarly the line (7) passes through the points (0, 40) and (80, 0) and the half plane is towards to origin.

The line (8) passes through the points (0, 20) and (10, 0) and the half plane is away from the origin.

Solving (6) and (7), we get $x_1 = 20, x_2 = 30$.

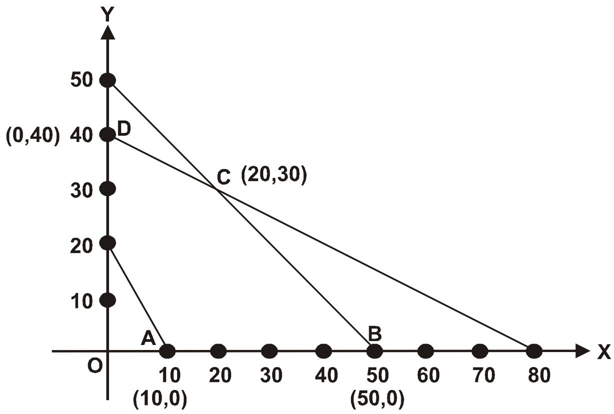
The lines (6) and (7) intersect at (20, 30).

The graph is shown in the figure.

the feasible region is

ABCDE where A is (10, 0), B (5, 0), C (20, 30), D(0, 40), E (0,20).

The value of Z at the extreme points are shown in the table.



Point	x_1	x_2	$z = 4x_1 + 3x_2$
A	10	0	z = 4 x 10 + 3.0 = 40
B	50	0	z = 4 x 50 + 3.0=200 (Maximum)
C	20	30	z = 4 x 20 + 3x30=80+90=170
D	0	40	z = 4 x 0 + 3 x 40 = 120
E	0	20	z = 4 x 0 + 3 x 20 = 60

The maximum value of z is 200.

It is obtained when x_1=50, x_2=0.

9.a) The given determinant

$$\Delta = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} (b+c)^2-2bc & a^2 & bc \\ (c+a)^2-2ca & b^2 & ca \\ (a+b)^2-2ac & c^2 & ab \end{vmatrix} \quad C_1 \to C_1 - 2C_3$$

$$= \begin{vmatrix} b^2+c^2 & a^2 & bc \\ c^2+a^2 & b^2 & ca \\ a^2+b^2 & c^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix} \quad (C_1 \to C_1 + C_2)$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2-a^2 & ca-bc \\ 0 & c^2-a^2 & ab-bc \end{vmatrix}$$

$(R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$

$$= (a^2+b^2+c^2)\begin{vmatrix} b^2-a^2 & ca-bc \\ c^2-a^2 & ab-bc \end{vmatrix}$$

$$= (a^2+b^2+c^2)\begin{vmatrix} -(a-b)(a+b) & c(a-b) \\ (c-a)(c+a) & -b(c-a) \end{vmatrix}$$

$$= (a^2+b^2+c^2)(a-b)(c-a)\begin{vmatrix} -(a+b) & c \\ c+a & -b \end{vmatrix}$$

$$= (a-b)(c-a)(a^2+b^2+c^2)$$

$$[b(a+b)-c(c+a)]$$

$$= (a-b)(b-c)(c-a)(a+b+c)(a^2+b^2+c^2)$$

(b) We know

$$(1+x)^n = C_0 + C_1x + c_2x^2 + c_3x^3 + \ldots + C_nx^n \quad \ldots(1)$$

Differentiating both sides w.r.t. x, we get

$$n(1+x)^{n-1} = C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$$

$$\Rightarrow C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1}$$

$$= n(1+x)^{n-1} \quad \ldots(2)$$

Again $\dfrac{(1+x)^n}{x^n} = C_0 + C_1.\dfrac{1}{x} + C_2.\dfrac{1}{x^2} +$

$$C_3.\frac{1}{x^3} + \ldots + C_n.\frac{1}{x^n}$$

$$\Rightarrow \frac{(1+x)^n}{x^n} = C_0 + C_1.\frac{1}{x} +$$

$$C_2.\frac{1}{x^2} + C_3.\frac{1}{x^3} + \ldots + C_n\frac{1}{x^n}$$

$$\Rightarrow C_0 + C_1\frac{1}{x} + C_2\frac{1}{x^2} + C_3.\frac{1}{x^3} + \ldots$$

$$+ C_n.\frac{1}{x^n} = \frac{(1+x)^n}{x^n} \quad \ldots(3)$$

Multiplying (2) and (3), we get

$$(C_1 + 2C_2x + 3C_3x^2 + \ldots + nC_nx^{n-1})$$

$$(C_0 + C_1.\frac{1}{x} + C_2.\frac{1}{x^2} + C_3.\frac{1}{x^3} + \ldots + C_n.\frac{1}{x^n})$$

$$= n(1+x)^{n-1}\frac{(1+x)^n}{x^n}$$

$$= n\frac{(1+x)^{2n-1}}{x^n}$$

$$= n.\frac{1}{x^n}[1 + {}^{2n-1}C_1 + {}^{2n-1}C_2x^2 +$$

$$= n.\frac{1}{x^n}[1 + {}^{2n-1}C_1 + {}^{2n-1}C_2x^2 +$$

$...{}^{2n-1}C_{n-1}x^{n-1}+...+x^{2n-1}]$

Equaling the coefficient of $\frac{1}{x}$ from both sides, we get

$$C_1^2+2C_2^2+3C_3^2+...+nC_n^2=n.{}^{2n-1}C_{n-1}$$

$$=n.\frac{(2n-1)!}{(n-1)!\,n!}$$

$$=\frac{n(2n-1)!}{(n-1)!\,n(n-1)!}$$

$$=\frac{(2n-1)!}{[(n-1)!]^2}$$

(c) (i) Let P(E) and P(H) be the probability of passing English and Hindi respectively.

Give that probability of passing both English and Hindi - 0.5

$P(E\cap H)=0.5$

Probability of passing neither = 0.1

$1-P(E\cup H)=0.1$

$\Rightarrow P(E\cup H)=1-0.1=0.1$

Given that the probability of passing English = 0.75

$P(E)=0.75$

We know

$P(E\cup H)=P(E)+P(H)-P(E\cap H)$

$\Rightarrow 0.9=0.75+P(H)-0.5$

$\Rightarrow 0.25+P(H)=0.9$

$\Rightarrow P(H)=0.9-0.25=0.65$

Probability of passing Hindi = 0.65

(ii) Given that $P(A)=0.4,\ P(B/A)=0.3$

$P(B^C/A^C)=0.2$

Here P(B / A) = 0.3

$$\Rightarrow \frac{P(B\cap A)}{P(A)}=0.3$$

$$\Rightarrow \frac{P(B\cap A)}{0.4}=0.3$$

$\Rightarrow P(A\cap B)=0.3\,x\,0.4=0.12$

Again given that $P(B^C/A^C)=0.2$

$$\Rightarrow \frac{P(B^C\cap A^C)}{P(A^C)}=0.2$$

$$\Rightarrow \frac{1-P(B^C\cap A^C)^C}{1-P(A)}=0.2$$

$$\Rightarrow \frac{1-P(B\cap A)}{1-0.4}=0.2$$

$$\Rightarrow \frac{1-P(A\cup B)}{0.6}=0.2$$

$\Rightarrow P(A\cup B)=1-0.6\,x\,0.2=1-0.12=0.88$

$\Rightarrow P(A\cup B)=0.88$

We know

$\Rightarrow P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$\Rightarrow 0.88=0.4+P(B)-0.12$

$=P(B)+0.28$

$\therefore P(B)=0.88-0.28=0.60$

$$P(A/B)=\frac{P(A\cap B)}{P(B)}=\frac{0.12}{0.60}=\frac{1}{5}.$$

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GROUP - A

OBJECTIVE AND VERY SHORT TYPE QUESTIONS

Each Question carries 1 marks

1. If $A = \{1, 2, 3, 4, 5, 6\}$ and a relation R on A is defined by R = {a, b) : a, b $\in$ A and a divides b}, then write R is Roster form.

2. Write the equivalence class $[3]_7$ as a set.

3. Sets A and B have respectively m and n elements. The total number of relations from A to B is 64. If m < n and m $\neq$ 1, write the values of m and n respectively.

4. If $A = \{1,2,3,4,5\}$ and $R : A \to A$ is $\{(1,2),(2,3),(4,5),(3,3)\}$ then write $R^{-1}: A \to A$.

5. If R = {(a, a^3) : a is a prime number less than 5} be a relation, then find the range of R.

6. If $f : R \to R$ and $g : R \to R$ are given by $f(x)=8x^3$ and $g(x) = x^{\frac{1}{3}}$ then write fog.

7. If R be a relation on a finite set A having n elements, then what is the number of relations on A ?

8. If $f(x) = \text{Cos}(\log_e x)$ then what is

$$f(x) - f(y) - \frac{1}{2}\left[f(xy) + f\left(\frac{x}{y}\right)\right]?$$

9. Write the principal value of

$$\text{Sin}^{-1}\left(\text{Sin}\frac{2\pi}{3}\right).$$

10. What is the value of

$$2\,\text{Sin}^{-1}\frac{1}{2} + \text{Cos}^{-1}\frac{1}{2}?$$

11. Find the value of

$$\text{Sin}\left[\text{Cot}^{-1}\{\tan(\text{Cos}^{-1}x\}\right].$$

12. Write the principal value of

$$\text{Sin}^{-1}\left(-\frac{1}{2}\right) + \text{Cos}^{-1}\text{Cos}\left(-\frac{\pi}{2}\right).$$

13. If $\tan^{-1}n + \tan^{-1}y = \frac{\pi}{4}, xy < 1,$ then write the value of x + y + xy.

14. If $\text{Sin}^{-1}\frac{x}{5} + \text{Cosec}^{-1}\left(\frac{5}{4}\right) = \frac{\pi}{2}$, then what is the value of x ?

15. Write the principal value of

$$\text{Cos}^{-1}\left(\text{Cos}\frac{2\pi}{3}\right) + \text{Sin}^{-1}\left(\text{Sin}\frac{2\pi}{3}\right).$$

16. If $\text{Sin}^{-1}\left(\frac{2a}{1+a^2}\right) - \text{Cos}^{-1}\left(\frac{1-b^2}{1+b^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right)$

then what is x ?

17. Find the value of

$$\tan^{-1}\left(\frac{yz}{xr}\right) + \tan^{-1}\left(\frac{xz}{yr}\right) + \tan^{-1}\left(\frac{xy}{zr}\right)$$

where $r^2 = x^2 + y^2 + z^2$.

18. Using the principal values, find the value of

$$\text{Cos}^{-1}\left(\text{Cos}\frac{13\pi}{6}\right).$$

19. What is the value of $2\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{1}{7}$.

20. Write the maximum value of x + y subject to $2x + 3y \le 6, x \ge 0, y \ge 0$.

21. Shade the region $2x + 3y \le 6, x \ge 0, y \ge 0$.

22. Write the solution of the following L.P.P.

Maximize $Z = 2x + 3y$

Subject to $x + y \le 1$, x, y ≥ 0

23. State the feasible solution.

24. Mention the quadrant in which the solution of an L.P.P. with two decision variables lie when the graphical method is adopted.

25. What are the values of x and y if

$\begin{vmatrix} x & y \\ 1 & 1 \end{vmatrix} = 2$ and $\begin{vmatrix} x & 3 \\ y & 2 \end{vmatrix} = 1$?

26. Determine the maximum value of

$\begin{vmatrix} \text{Cos}\,x & \text{Sin}\,x \\ -\text{Sin}\,x & \text{Cos}\,x - 1 \end{vmatrix}$.

27. What is the value of $\begin{vmatrix} -\text{Cosec}^2\theta & \text{Sec}^2\theta & -0.2 \\ \text{Cot}^2\theta & -\tan^2\theta & 1.2 \\ -1 & 1 & 1 \end{vmatrix}$?

28. What is the value of $\begin{vmatrix} \frac{1}{a} & 1 & bc \\ \frac{1}{b} & 1 & ca \\ \frac{1}{c} & 1 & ab \end{vmatrix}$?

29. What is the value of $\begin{vmatrix} b+c & a-b & a \\ c+a & b-c & b \\ a+b & c-a & c \end{vmatrix}$?

30. If ω is a complex root of 1 then for what value of λ, the dterminant

$\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \lambda & 1 \\ \omega^2 & 1 & \omega \end{vmatrix} = 0$?

31. What is the value of $\begin{vmatrix} 0 & 8 & 0 \\ 25 & 520 & 25 \\ 1 & 410 & 0 \end{vmatrix}$?

32. Find the value of $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix}$.

33. If $\begin{bmatrix} 3 & 5 & 3 \\ 2 & 4 & 2 \\ \lambda & 7 & 8 \end{bmatrix}$ is a singular matrix then write the value of λ.

34. If $\begin{bmatrix} 3 & 2 \\ 7 & x \end{bmatrix}\begin{bmatrix} 5 & -2 \\ -7 & y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

then write the value of x and y.

35. If A is a square matrix of order 3 and $|A| = 3$, then write the matrix represented by A(adj A).

36. If $3A - B = \begin{bmatrix} 5 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 \\ 2 & 5 \end{bmatrix}$ then find the matrix A.

37. If, $a_{i_j} = |i - j|$, then construct $\left[a_{i_j}\right]_{2\times 3}$.

38. If $\begin{bmatrix} 9 & -1 & 4 \\ -2 & 1 & 3 \end{bmatrix} = A + \begin{bmatrix} 1 & 2 & -1 \\ 0 & 4 & 9 \end{bmatrix}$

then find the matrix A.

39. If $\begin{bmatrix} 2x & 1 \\ 5 & x+2y \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 5 & 0 \end{bmatrix}$ then find the value of x + y.

40. Find x from the matrix equation

$\begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}\begin{bmatrix} x \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ then find the value of x.

41. If $2P(A) = P(B) = \frac{5}{13}$ and $P(A/B) = \frac{2}{5}$ then what is $P(A \cup B)$?

42. A binomial distribution has mean 4 and variance 3, write the number of trials.

43. If an event A is independent to itself, then what is P(A).

44. If E and F are events such that P(F)=0.3 and $P(E \cap F) = 0.2$.

Find P(E/F).

45. If A and B are independent events and

$P(A) = \frac{3}{5}, P(B) = \frac{1}{5}$, what is $P(A \cap B)$?

46. If P(B)=0.5 and $P(A \cap B) = 0.32$ then find P(A/B).

47. If A and B are two events such that P(A/B)=P(B/A) then what is the relation between P(A) and P(B).

48. Define conditional probability.

49. If $f(x)=\dfrac{x^3+x^2-16x+20}{(x-2)^2}$ if $x\neq 2$

if = k if x = 2

is continous for all x, then what is the value of k?

50. If $f(x)=(1+2x)^{\frac{1}{x}},\ x\neq 0$ is continuous at x=0, then what is the value of f(0).

51. If a function f is continuous at x=a, then what is $\lim\limits_{h\to 0^+}\frac{1}{2}[f(a+h)+f(a-h)]$.

52. What is the derivative of $\operatorname{Sec}^{-1}\left(\dfrac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$?

53. If $y=\operatorname{Sec}^{-1}\left(\dfrac{x+1}{x-1}\right)+\operatorname{Sin}^{-1}\left(\dfrac{x-1}{x+1}\right)$, then find $\dfrac{dy}{dx}$.

54. Write the derivative of Sin x with respect to Cos x ?

55. If $f(0)=0,\ f^1(0)=2$, then what is the derivative of y = f (f (f (f (x)))) at x = 0.

56. Write the condition of the Role's theorem which is violated by the function $f(x)=|x-1|$ in [0,2].

57. Mention the values of x for which the function $f(x)=x^3-12x$ is decreasing.

58. What is the acceleration at the end of 2 seconds of the particle that moves with the rule $s=\sqrt{t}+1$?

59. What is the rate of change of the area of a circle with respect to its radius ?

60. What is the slope of the normal to the curve $2y = 3 - x^2$ at the point (1, 1)?

61. Write the equation of the tangent to the curve $y=|x|$ at the point (-2, 2).

62. What is the slope of the normal to the curve $x^{\frac{2}{3}}+y^{\frac{2}{3}}=20$ at the point (8, 64) ?

63. For which value of x, the function $f(x)=3x^2-x+3$ is minimum?

64. For which value of x the function $f(x)=4-x-x^2$ is maximum?

65. Write the definite integral which is equal to

$$\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\frac{r}{\sqrt{n^2+r^2}}.$$

66. What is the value of $\int_0^{\pi/2}\log\tan x\,dx$?

67. What is the value of $\int_0^{\pi/2}\dfrac{f(x)}{f(x)+f\left(\frac{\pi}{2}-x\right)}dx$.

68. Write the value of

$$\int_0^{\pi/2}\frac{\operatorname{Sin}x}{\operatorname{Sin}x+\operatorname{Cos}x}dx-\int_0^{\pi/2}\frac{\operatorname{Cos}x}{\operatorname{Sin}x+\operatorname{Cos}x}dx$$

69. Evaluate $\int_0^1[3x]\,dx$.

70. Integrate $\int\dfrac{\operatorname{Sin}6x+\operatorname{Sin}4x}{\operatorname{Cos}6x+\operatorname{Cos}4x}dx$.

71. Integrate $\int\dfrac{dx}{x[(\operatorname{Log}x)^2+25]}$.

72. Evaluate $\int_0^{\pi/2}\dfrac{\sqrt{\operatorname{Sin}x}}{\sqrt{\operatorname{Sin}x}+\sqrt{\operatorname{Cos}x}}dx$.

73. What is the value of $\int_0^{\pi/4}\operatorname{Cos}^4x\operatorname{Sin}^{99}x\,dx$.

74. Find $\int\left[\sqrt{a^2-x^2}+\dfrac{x^2}{\sqrt{a^2-x^2}}\right]dx$.

75. Write the value of $\int_{-\pi/3}^{\pi/3}\left[x^4 \operatorname{Sin}x^3 + x\operatorname{Cos}x^2\right]dx$.

76. What is the value of $\int_1^3 \tan^{-1}x\,dx + \int_1^3 \operatorname{Cot}^{-1}x\,dx$?

77. $\int e^{\ln(\operatorname{Cosec}^2 x - \operatorname{Cot}^2 x)\,dx} = ?$

78. What is the area bounded by $x = e^y$, x = 0, y = 0 and y = 1 ?

79. What is the area bounded by y = x, x = 0 and y = 1 ?

80. Write the area bounded by y = -2x, y = 0, x=1 and x = 3.

81. If p and q are the order and degree of the differential equation

$$y\left(\frac{dy}{dx}\right)^2 + x^2\frac{d^2y}{dx^2} + xy = \operatorname{Sin}x$$

then what are the values of p and q.

82. Find the particular solution of the differential equation $\frac{d^2y}{dx^2} = 6x$ given that y=1 and $\frac{dy}{dx} = 2$ when x = 0.

83. Solve the differential equation $(x + 2y^3)\frac{dy}{dx} = y$.

84. Solve $\frac{d^2y}{dx^2} = \frac{1}{x(x+1)} + \operatorname{Cosec}^2 x$.

85. Obtain the differential equation whose solution is $y = Ae^{2x} + Be^{-2x}$.

86. Write the particular solution of

$$\frac{dy}{dx} = (1+x)^4, \text{ y = 0 when x = -1.}$$

87. Given the general solution as $y = (x^2 + c)e^{-x}$ of a diffeential equation what is the particular solution if y= 0 when x = 1.

88. What is the differential equation whose general solution is y = 3x + k.

89. Write the values of m and n for which the vectors $(m-1)\hat{i} + (n+2)\hat{j} + 4\hat{k}$ and $(m+1)\hat{i} + (n-2)\hat{j} + 8\hat{k}$ will be parallel.

90. Prove that $|\vec{a} + \vec{b}| \le |\vec{a}| + |\vec{b}|$. Write when equality will hold.

91. If $(\vec{a} \text{ x } \vec{b})^2 + (\vec{a}.\vec{b})^2 = 144$, then write the value of ab.

92. What is the angle between $\hat{i} + \hat{j}$ and $\hat{i} - \hat{j}$?

93. If $|\vec{\alpha} + \vec{\beta}| = |\vec{\alpha} - \vec{\beta}|$ then what is the angle between $\vec{\alpha}$ and $\vec{\beta}$?

94. Write the vector equation of the plane whose cartesian form is 3x - 4y + 2z = 5.

95. Write the distance between paralel planes 2x - y + 3z = 4 and 2x - y + 3z - 18 = 0.

96. Write the angle between the planes 3x - 5y + 2z - 8 = 0 and 2x + 4y + 7z + 16 = 0.

97. What are the direction cosines of the line perpendicular to the plane 3x - 2y - 2z + 1 = 0.

98. Find the value of k for which the line

$$\frac{x-2}{3} = \frac{1-y}{k} = \frac{z-1}{4}$$

is parallel to the plane 2x + 6y + 3 z - 4 = 0.

99. Write the equation of the line passing through (-3,1,2) and perpendicular to the plane 2y-z=3.

100. If the direction cosines of a straight line be $\left\langle \frac{2}{7}, \frac{3}{7}, \frac{k}{7} \right\rangle$ then what is the value of k ?

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GROUP - A

ANSWERS

1. R = {(a, b) : a, b ∈ A and a divides b}
   = {(1, 1), (1, 2), (1,3), (1,4), (1,5), (1,6), (2,2), (2,4), (2,6), (3,3), (3,6), (4,4), (5,5), (6,6) }

2. $[3]_7 = \{.... -11, -4, 3, 10, 17, 24, ...\}$

3. m = 2, n = 3

4. $R^{-1} = \{2,1), (3,2), (5,4), (3,3)\}$

5. $R = \{(2, 2^3), (3, 3^3)\}$
   $= \{(2,8), (3,27)\}$
   Range of R = {8, 27}.

6. $f(x) = 8x^3, g\ (x) = x^{\frac{1}{3}}$
   $(fog)(x) = f\,[g(x)] = f\left(x^{\frac{1}{3}}\right) = 8\,(x^{\frac{1}{3}})^3 = 8x$

7. $|A| = n,$
   The number of elements of A x A is n x n = n^2.
   The number of relations $= 2^{n^2}$.

8. 0

9. $\text{Sin}^{-1}\left(\text{Sin}\frac{2\pi}{3}\right) = \text{Sin}^{-1}\left[\text{Sin}\left(\pi - \frac{\pi}{3}\right)\right]$
   $= \text{Sin}^{-1}\ \text{Sin}\frac{\pi}{3} = \frac{\pi}{3}$
   Required Principal Value $= \frac{\pi}{3}$.

10. $2\ \text{Sin}^{-1}\frac{1}{2} + \text{Cos}^{-1}\frac{1}{2} = 2.\frac{\pi}{6} + \frac{\pi}{3} = \frac{2\pi}{3}$.

11. x.

12. $\text{Sin}^{-1}\left(-\frac{1}{2}\right) + \text{Cos}^{-1}\left[\text{Cos}\left(-\frac{\pi}{2}\right)\right]$
    $= -\frac{\pi}{6} + \text{Cos}^{-1}\ \text{Cos}\frac{\pi}{2} = -\frac{\pi}{6} + \text{Cos}^{-1}0 = -\frac{\pi}{6} + 1.$

13. x + y + xy = 1

14. x = 3

15. The principal value branch of Sin^{-1} is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
    and for Cos^{-1} is $[0, \pi]$.
    $\therefore\ \text{Cos}^{-1}\left(\text{Cos}\frac{2\pi}{3}\right) + \text{Sin}^{-1}\,\text{Sin}\frac{2\pi}{3}$
    $= \frac{2\pi}{3} + \text{Sin}^{-1}\,\text{Sin}\left(\pi - \frac{\pi}{3}\right)$
    $= \frac{2\pi}{3} + \text{Sin}^{-1}\,\text{Sin}\frac{\pi}{3} = \frac{2\pi}{3} + \frac{\pi}{3} = \pi$

16. $x = \frac{a-b}{1+ab}$

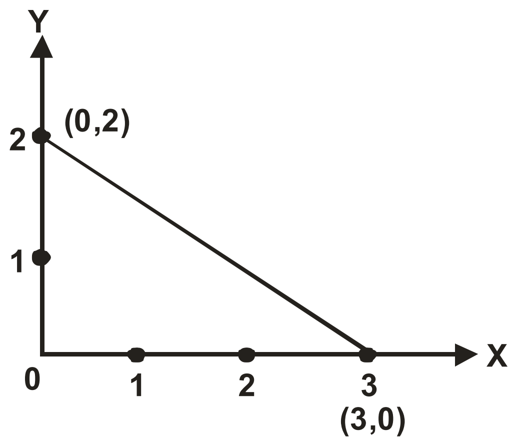
17. $\frac{\pi}{2}$

18. $\text{Cos}^{-1}\,\text{Cos}\left(\frac{13\pi}{6}\right) = \text{Cos}^{-1}\left[\text{Cos}\left(2\pi + \frac{\pi}{6}\right)\right]$
    $= \text{Cos}^{-1}\,\text{Cos}\left(\frac{\pi}{6}\right) = \frac{\pi}{6}$.

19. $\frac{\pi}{4}$

20. The maximum value of Z is 3 which is obtained when x = 3, y = 0.

21.



22. Maximum value of Z is 3 and is obtained when x = 0, y = 1.

23. Any solution of the general L.PP which also satisfies the non-negative restriction is called a feasible solution.

24. In the graphical solution, the solution of a L.P.P. lies on the first quadrants.

25. x = 5, y= 3

26. 0

27. 0

28. 0

29. $-(a^3+b^3+c^3-3abc)$

30. λ is any number.

31. 200

32. 0

33. -33

34. x = 5, y = 3

35. $A(\text{adj}A)=\begin{bmatrix}3&0&0\\0&3&0\\0&0&3\end{bmatrix}$

36. $A=\begin{bmatrix}3&1\\1&2\end{bmatrix}$

37. $A=\begin{bmatrix}0&1&2\\1&0&1\end{bmatrix}$

38. $A=\begin{bmatrix}8&-3&5\\-2&-3&-6\end{bmatrix}$

39. x + y = 1

40. x = -1

41. $2P(A)=\frac{5}{13}$,

$\Rightarrow P(A)=\frac{5}{26}$

Given that $=P(B)=\frac{5}{13}$

Also $P(A/B)=\frac{2}{5}$

$\Rightarrow \frac{P(A\cap B)}{P(B)}=\frac{2}{5}$

$\Rightarrow \frac{P(A\cap B)}{5/13}=\frac{2}{5}$

$\Rightarrow P(A\cap B)=\frac{5}{13}\cdot\frac{2}{5}=\frac{2}{13}$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

$=\frac{5}{26}+\frac{5}{13}-\frac{2}{13}=\frac{11}{26}$

42. Let n be the number of trials.

Let p be the probability of an event A.

$\therefore p=1-q$

We known $\mu=np=4$ ... (1)

Variance $\upsilon=\sigma^2=npq=3$ ... (2)

$\frac{npq}{np}=\frac{3}{4}\Rightarrow q=\frac{3}{4}$

$p=1-q=1-\frac{3}{4}=\frac{1}{4}$

From (2), we have

$n.\frac{1}{4}.\frac{3}{4}=3$

$\Rightarrow n=16$.

So the number of trails = 16.

43. P(A) = 0 or 1.

44. $P(E/F)=\frac{2}{3}$.

45. $P(A \cap B) = \frac{3}{25}$

46. $P(A/B) = \frac{16}{25}$

47. P(A) = P(B)

48. If B be an event in a sample space then the conditional probability of B subject to A which is written as P(B/A) is defined as

$$P(B/A) = \frac{P(B \cap A)}{P(A)},\ P(A) \neq 0.$$

49. k = 7

50. f(0) = e^2

51. f(a)

52. $\frac{2}{x}$

53. 0

54. - Cot x

55. 16

56. Given function is $f(x) = |x-1|$ in [0, 2]

$$\Rightarrow f(x) = \begin{cases} x-1 & \text{when } x \geq 1 \\ -(x-1) & \text{when } n < 1 \end{cases}.$$

The function is continuous in [0, 2].

At x = 1, L.H.D. $= \lim_{x\to1^-} \frac{f(x)-f(1)}{x-1}$

$$= \lim_{x\to1^-} \frac{-(x-1)-0}{x-1} = -1$$

R. H. D. $= \lim_{x\to1^+} \frac{f(x)-f(1)}{x-1}$

$$= \lim_{x\to1^+} \frac{x-1-0}{x-1} = 1$$

At x = 1, L.H.D. $\neq$ R.H.D.

The function is not differentiable at x =1.

Again $f(0) = |0-1| = 1$

$f(2) = |2-1| = 1$

In order that the Rolle's theorem is verified

(i) the function should be continuous in [0,2]

(ii) the function is differentiable in (0, 2).

(iii) f(0)=f(2).

Here the 2nd condition of Rolle's theorem is violated.

57. The function is decreasing $x \geq 2$ for and $x \leq -2$.

58. $-\frac{1}{8\sqrt{2}}$ units/sec^2

59. $2\pi r$

60. 1

61. x + y = 0

62. $\frac{1}{2}$

63. The function is minimum at $x = \frac{1}{6}$.

64. The function is maximum at $x = -\frac{1}{2}$.

65. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \frac{r}{\sqrt{n^2+r^2}} = \int_0^1 \frac{x}{1+x^2}dx$

66. $\int_0^{\pi/2} \log \tan x\, dx = 0$

67. $\frac{\pi}{4}$

68. 0

69. 1

70. $\frac{1}{5}$ log Sec 5 x + C

71. $\frac{1}{5}\tan^{-1}\left(\frac{1}{5}\log x\right)+C$

72. $\frac{\pi}{4}$

73. 0

74. $a^2 \text{Sin}^{-1}\frac{x}{a}+C$

75. 0

76. π

77. x + C

78. -8 sf units.

79. $\frac{1}{2}$ sf. unit

80. -8 sf. units

81. p = 2, q = 1

82. $y = x^3 + 2x + 1$

83. $\frac{x}{y} = y^2 + C$

84. y = x lnx - x ln (x+1) - ln (x+1)

- ln Sin x + cx+D.

85. $\frac{d^2y}{dx^2} = 4y$

86. $y = \frac{(1+x)^5}{5}$

87. Particular solution is $y = (x^2+1)e^{-x}$.

88. $\frac{dy}{dx} = 3$

89. m = 3, n = -6

90. Let $\vec{OA} = \vec{a}$, $\vec{AB} = \vec{b}$

$\therefore \vec{OB} = \vec{OA} + \vec{AB} = \vec{a} + \vec{b}$.

We know $|\vec{OB}| \le |\vec{OA}| + |\vec{AB}|$

$\Rightarrow |\vec{a}+\vec{b}| \le |\vec{a}| + |\vec{b}|$

Let $\vec{b}$ be in the directions of $\vec{a}$.

$\therefore |\vec{OB}| = |\vec{OA}| + |\vec{AB}|$

$\Rightarrow |\vec{a}+\vec{b}| = |\vec{a}| + |\vec{b}|$.

Thus equality holds when $\vec{a}$ and $\vec{b}$ are in the same direction.

91. ab = 12

92. $\theta = 90^0$

$\therefore$ The angle between two vectors = 90^0.

93. The angle between $\vec{\alpha}$ and $\vec{\beta}$ is 90^0.

94. $\vec{r} \cdot (3\hat{i} - 4\hat{j} + 2\hat{k}) = 5$.

95. $\sqrt{14}$

96. $\theta = 90^0$

97. $< \frac{3}{\sqrt{17}}, \frac{-2}{\sqrt{17}}, \frac{-2}{\sqrt{-17}} >$

98. k = 3

99. $\frac{x+3}{0} = \frac{y-1}{2} = \frac{z-2}{-1}$

100. $k = \pm 6$

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GROUP - B

SHORT TYPE QUESTIONS

Each questions carries 4 marks

2. Relation and Functions, Inverse Trigonometric functions, Linear Programming

1. Show that the relation R defined on the Set Z of all integers defined as R = {(x, y) : x - y is an integer} is reflexive sytmmetric and transitive.

2. It R and S are two equivalence relations on a set then prove that $R \cap S$ is also an equivalence relation on the set.

3. Prove that $f : R \to R$ such that $f(x) = \frac{2x^2}{x^2+1}$ is neither on to nor one-one function.

4. If $f : R \to R$ is a function defined by $f(x) = 4x^3 + 7$, the show that f is a bijection.

5. If $f(x) = \sqrt{x}$ and $g(x) = 1 - x^2$, then compute fog and gof and find their natural domain.

6. Let f be a real function. Show that f(x) + f(-x) is always an even function and f(x) - f(-x) is always an odd function.

7. Construct an example to show that $f(A \cap B) \neq f(A) \cap f(B)$ where $A \cap B \neq \phi$.

8. Prove that $f : x \to y$ is injective iff $f^{-1}(f(A)) = A$ for all $A \subseteq X$.

9. Prove that $f : x \to y$ is injective iff for all subjects A, B of X, $f(A \cap B) = f(A) \cap f(B)$.

10. If f and g are functions on R given by f(x)=Sin x, g(x) = x^5 then find the composition f∘g and g∘f. Test whether f∘g = g∘f.

11. Show that the relation R on the set R of real numbers defined as $R = \{(a,b) : a \leq b^2\}$ is neither reflexive, nor symmetric nor transitive.

12. If m and n are integers and f(m, n) is defined by $f(m,n) = \{^{5 \text{ if } m<n}_{f(m-n,\ n+2)+n \text{ if } m \geq n}$ then find f(5, 3).

13. If S is a set of all rational numbers except 1 and * be defined on S by a * b = a + b - ab for all $a, b \in S$, then prove that
    (i) * is a binary operation on S.
    (ii) * is commutative as well as associative.

14. Consider the binary operation * on the set {1,2,3,4,5} defined by a * b = min {a, b}. Write the operation table of operation *

15. If * is the binary operation on N given by a * b = L.C.M. of a and b. Find 20 * 16. Is * (i) Commutative and (ii) associative.

16. Show that $\text{Sin}^{-1}\frac{4}{5} + 2\tan^{-1}\frac{1}{3} = \frac{\pi}{2}$.

17. Prove that $\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9} = \frac{1}{2}\tan^{-1}\frac{4}{3}$.

18. Prove that $\tan^{-1}\frac{3}{4} + \tan^{-1}\frac{3}{5} - \tan^{-1}\frac{8}{19} = \frac{\pi}{4}$.

19. If $\text{Cos}^{-1}x + \text{Cos}^{-1}y + \text{Cos}^{-1}z = \pi$ then prove that $x^2 + y^2 + z^2 + 2xyz = 1$.

20. If $\text{Sin}^{-1}x + \text{Sin}^{-1}y + \text{Sin}^{-1}z = \pi$ then show that $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2} = 2xyz$.

21. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$, then show that x + y + z = xyz.

22. Show that $\tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right) = -\frac{7}{17}$.

23. If $u = \cot^{-1}(\sqrt{\cos\alpha}) - \tan^{-1}(\sqrt{\cos\alpha})$ then prove that $\sin u = \tan^2\frac{\alpha}{2}$.

24. Prove that

$$\sin^{-1}\sqrt{\frac{x-q}{p-q}}=\cos^{-1}\sqrt{\frac{p-x}{p-q}}=\tan^{-1}\sqrt{\frac{x-q}{p-x}}$$

25. Prove that

$\tan^{-1}x+\cot^{-1}(x+1)=\tan^{-1}(x^2+x+1)$.

26. Show that $4\left(\cot^{-1}\frac{3}{2}+\operatorname{cosec}^{-1}\sqrt{26}\right)=\pi$.

27. In a Δ ABC, $<A=90$; then prove that $\tan^{-1}\left(\frac{b}{a+c}\right)+\tan^{-1}\left(\frac{c}{a+b}\right)=\frac{\pi}{4}$ when a, b, c are the sides of a triangle.

28. Prove that

$$\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right).$$

29. Prove that

$$\cos[\tan^{-1}\{\sin(\cot^{-1}x)\}]=\sqrt{\frac{1+x^2}{2+x^2}}.$$

30. Prove that $\tan^{-1}\sqrt{x}=\frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

when $x\in(0,1)$.

31. Prove that $\sec^2(\tan^{-1}3)-\operatorname{cosec}^2(\cot^{1}3)=0$.

32. Solve $\cos(\tan^{-1}x)=\sin\left(\cot^{-1}\frac{3}{4}\right)$.

33. Solve $\tan^{-1}\left(\frac{1-x}{1+x}\right)=\frac{1}{2}\tan^{-1}x$.

34. Solve $\tan^{-1}x+2\cot^{-1}x=\frac{2\pi}{3}$.

35. Prove that $\cos^{-1}x+\cos^{-1}\left[\frac{x}{2}+\frac{\sqrt{3-3x^2}}{2}\right]=\frac{\pi}{3}$.

36. Solve for x.

$$\tan^{-1}(x+2)+\tan^{-1}(x-2)=\tan^{-1}\left(\frac{8}{79}\right).$$

37. Solve for x

$$\tan^{-1}\left(\frac{1+x}{1-x}\right)=\frac{\pi}{4}+\tan^{-1}x, 0<x<1.$$

38. Solve for x, $\cos(2\sin^{-1}x)=\frac{1}{9}$, $x>0$.

39. Find the feasible region of the following system

$2x+y\geq 6,\ x-y\leq 3,\ x\geq 0,\ y\geq 0$.

40. Solve the following L.P.P.

Maximize $Z=20x+30y$

Subject to $3x+5y\leq 5$

$x, y, \geq 0$.

41. Solve the following L.P.P.

Minimize $Z=6x_1+7x_2$

Subject to $x_1+2x_2\geq 0$

$x_1,\ x_2\geq 0$.

42. Let a L.P.P. be

Maximize $Z=3x_1+5x_2$

Subject to $5x_1+3x_2\leq 30$

$x_1+2x_2\leq 12$

$2x_1+5x_2\leq 20$.

Test whether the point and $x_1,\ x_2\geq 0$ (2, 3) & (-3, 4) are feasible solutions or not.

43. Solve the following L.P.P.

Minimize $z=5x+7y$

Subject to $2x+y\geq 8$

$x+2y\geq 10$

$x\geq 0,\ y\geq 0$.

44. Solve the L.P.P.

Maximize $z=5x+3y$

Subject to $3x+5y\leq 15$

$5x+2y\leq 10$

$x\geq 0,\ y\geq 0$.

45. Solve the L.P.P.

Maximize $z=3x+2y$

Subject to $x+y\leq 400$

$2x+y\leq 500$

$x\geq 0,\ y\geq 0$.

3. Matrices, Determinants & Probability

1. If $A=\begin{bmatrix}1 & -2 & 2\\ 3 & 1 & 1\end{bmatrix}$ and $B=\begin{bmatrix}2 & 4\\ 1 & 2\\ 3 & -1\end{bmatrix}$

   then verify that $(AB)^T = B^T A^T$.

2. If $A=\begin{bmatrix}3 & -4\\ 1 & -1\end{bmatrix}$ then show that

   $A^k=\begin{bmatrix}1+2k & -4k\\ k & 1-2k\end{bmatrix}$.

3. If $\begin{bmatrix}1 & -2\\ 3 & 2\end{bmatrix}\begin{bmatrix}x & 2\\ 1 & y\end{bmatrix}=\begin{bmatrix}-3 & 4\\ -1 & 4\end{bmatrix}$

   then find the value of x and y.

4. If $A=\begin{bmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{bmatrix}$ then show that $A^3 = A^2$.

5. If $A=\begin{bmatrix}3 & 1\\ -1 & 2\end{bmatrix}$

   then show that $A^2-5A+7I=0$.

6. Find the inverse of the matrix $\begin{bmatrix}2 & 5\\ 1 & 3\end{bmatrix}$.

7. Find x so that

   $$\begin{bmatrix}1 & x & 1\end{bmatrix}\begin{bmatrix}1 & 3 & 2\\ 0 & 5 & 1\\ 0 & 3 & 2\end{bmatrix}\begin{bmatrix}1\\ 1\\ x\end{bmatrix}=0.$$

8. If A, B, C are matrices of order 2 x 2 each and

   $2A+B+C=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}$, $A+B+C=\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}$

   and $A+B-C=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}$ then find A, B, C.

9. If $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$ then find the value

   of $A^2-3A+2I$.

10. If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$ the prove that

    $A^2-4A-5I=0$.

11. If the matrix A is such that

    $\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}A=\begin{bmatrix}-4 & 1\\ 7 & 7\end{bmatrix}$ then find A.

12. Find the adjant of the matrix $\begin{bmatrix}1 & 1 & 2\\ 0 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}$.

13. Find the matriex which when added to

    $\begin{bmatrix}2 & -3\\ -4 & 7\end{bmatrix}$ gives $\begin{bmatrix}4 & -1\\ 3 & 2\end{bmatrix}$.

14. Using elementary transformation find the inverse of the matrix $\begin{bmatrix}1 & 3\\ 2 & 7\end{bmatrix}$.

15. Using elementary operation find the inverse of $\begin{bmatrix}4 & 5\\ 3 & 4\end{bmatrix}$.

16. Prove without expanding that

    $$\begin{vmatrix}bc & a & a^2\\ ca & b & b^2\\ ab & c & c^2\end{vmatrix}=\begin{vmatrix}1 & a^2 & a^3\\ 1 & b^2 & b^3\\ 1 & c^2 & c^3\end{vmatrix}.$$

17. Show that a+1 is a factor of $\begin{vmatrix}a+1 & 2 & 3\\ 1 & a+1 & 3\\ 3 & -6 & a+1\end{vmatrix}$.

18. Show that

    $$\begin{vmatrix}a-b-c & 2a & 2a\\ 2b & b-c-a & 2b\\ 2c & 2c & c-a-b\end{vmatrix}=(a+b+c)^3.$$

19. Show that $\begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix} = 1+a^2+b^2+c^2$.

20. Factorise $\begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix}$.

21. Without expanding find the value of the determinanat $\begin{vmatrix} 3 & 6 & 9 \\ -2 & 4 & -6 \\ 8 & 16 & 24 \end{vmatrix}$.

22. Prove that $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$ is a perfect square.

23. Prove that $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$.

24. Solve for x, $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$.

25. Find the value of $\begin{vmatrix} 17 & 58 & 97 \\ 19 & 60 & 99 \\ 18 & 59 & 98 \end{vmatrix}$.

26. Prove that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right).$$

27. Prove that

$$\begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix} = 0$$

28. Prove that $\begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$.

29. Prove that

$$\begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix} = xyz(x-y)(y-z)(z-x)$$

30. Prove that $\begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

$= (a-b)(b-c)(c-a)(bc+ca+ab)$

31. If A and B are two events such that $P(A) = 0.6$ $P(B) = 0.5$ and $P(A \cap B) = 0.2$ then find $P(A/B^C)$.

32. A pass of dice in thrown. Find the probability of getting at least 9 if 5 appears on atleast one of dice.

33. If two dice are thrown and if A be the event that one of the dice is 3 and B be the event that sum 5 occurs then find P(A/B).

34. If A and B are two events such that P(A)=0.3, P(B)=0.4 and $P(A \cup B) = 0.6$ then find (i) $P(A/B^C)$ (ii) $P(B/A)$.

35. If A and B are two independent events and P(A)=0., P(B)=0.6, then find

(i) P (A and B) (ii) P(A and not B)

(iii) P(A or B) (iv) P (neither A nor B)

36. A random variable has the following probability distribution.

x	0	1	2	3	4	5	6	7
P(x)	0	2p	2p	3p	p^2	$2p^2$	$7p^2$	2p

What is p ?

37. Two different digits are selected at random from the digits 1 through 9. If the sum is even, what is the probability that 3 is one of the digits selected ?

38. If x follows a binomial distribution with parameter n = 6 and p with 4P(x = 4) = P(x = 2). Find p.

39. Find the mean of the numbr of heads in three tosses of a coin.

40. If P(A) = 0.6, P(B/A) = 0.5, then find $P(A \cup B)$ when A and B are independent.

41. Show that $\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix} = 0$.

42. Prove that $\begin{vmatrix} a^2 & bc & c^2+ac \\ a^2+ab & b^2 & ca \\ ab & b^2+bc & c^2 \end{vmatrix} = 0$.

43. If x = cy+bz, y = az+cx and z = bx+ay where x, y, z are not all zero then prove that $a^2+b^2+c^2+2abc = 1$, by determinant method.

44. Prove that

$$\begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & b^2+bc & c^2 \end{vmatrix} = 4(b+c)(c+a)(a+b).$$

45. Solve $\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$

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4. Continuity and Differentiability, Application of Derivatives

1. If $f(x)\begin{cases} ax^2+b & \text{if } x<1 \\ 1 & \text{if } x=1 \\ 2ax-b & \text{if } x>1 \end{cases}$

   is continuous at x=1, then find a and b.

2. Show that sin x is continuous for every real x.

3. Find the value of k if the function f(x) defined by is

   $f(x)\begin{cases} 2x-1 & \text{when } x<2 \\ k & \text{when } x=2 \\ x+1 & \text{when } x>2 \end{cases}$

   is continious at x = 2.

4. Find the value of k so that the function f defined by

   $f(x)\begin{cases} kx+1 & \text{if } x\le\pi \\ \cos x & \text{if } x>\pi \end{cases}$

   is continious at $x=\pi$.

5. Find the value of k so that the function 'f' defined below, is continuous at x=0 where

   $f(x)=\begin{cases} \dfrac{1-\cos 4x}{8x^2} & \text{if } x\ne 0 \\ k & \text{if } x=0 \end{cases}$.

4. If $x = a\cos^3\theta$ and $y = a\sin^3\theta$, then find $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{6}$.

7. If x sin (a+y) + sin a cos (a+y) = 0 then prove that $\dfrac{dy}{dx} = \dfrac{\sin^2(a+y)}{\sin a}$.

8. Find $\dfrac{dy}{dx}$ if $x^m.y^n = \left(\dfrac{x}{y}\right)^{m+n}$.

9. Find $\dfrac{dy}{dx}$ when $y = \sin^{-1}\left(\dfrac{2\sqrt{t}-1}{t^2}\right)$.

10. If $x = a \sec\theta$, $y = b\tan\theta$ then prove that $\frac{d^2y}{dx^2} = -\frac{b^4}{a^2y^3}$.

11. If $x^y = e^{x-y}$, then show that $\frac{dy}{dx} = \frac{\log x}{[\log(xe)]^2}$.

12. If sin y=x sin (a+y) then show that $\frac{dy}{dx} = \frac{\sin^2(a+y)}{\sin a}$.

13. If $x = a(\cos t + t\sin t)$, $y = a(\sin t - t\cos t)$ then find $\frac{d^2y}{dx^2}$.

14. If $\cos x = \sqrt{\frac{1}{1+t^2}}$, $\sin y = \frac{2t}{1+t^2}$ then show that $\frac{dy}{dx}$ is independent of t.

15. Find $\frac{dy}{dx}$ when $y^x = x^{\sin y}$.

16. If $y^2 \cot x = x^2 \cot y$ then find $\frac{dy}{dx}$.

17. Find the derivative of $x^{\sin x}$ with respect to x.

18. Differentiate $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$ with respect to $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$.

19. Differentiate $y = \tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$.

20. Differentiate $y = (\sin y)^{\sin 2x}$.

21. Test the differentiability and continuity of the following functioon at x = 0

$$f(x) = \begin{cases} \frac{1-e^{-z}}{x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

22. Differentiate $\sec^{-1}\left(\frac{1}{2x^2-1}\right)$ with respect to $\sqrt{1-x^2}$.

23. If $y = x + \cfrac{1}{x + \cfrac{1}{x + \ldots \text{to}\infty}}$ then find $\frac{dy}{dx}$.

24. Find the slope of the tangent to the curve $x = 2(\theta - \sin 2\theta)$, $y = 2(1-\cos\theta)$ at $\theta = \frac{\theta}{4}$.

25. If $\cos y = x\cos(a+y)$ then show that $\frac{dy}{dx} = \frac{\cos^2(a+y)}{\sin a}$.

26. If $y = \tan(x+y)$ then show that $\frac{dy}{dx} = -\frac{1+y^2}{y^2}$.

27. Find the derivative of $\sin^{-1}\left(\frac{2x^3}{1+x^6}\right)$.

28. Find the derivative of $\tan^{-1}(\cos^2 x)$.

29. Differentiate $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$.

30. Find $\frac{dy}{dx}$ when $y = x^y$.

31. If $2z = x\left(2 + \frac{dz}{dx}\right)$ then prove that $\frac{d^2z}{dx^2}$ is constant.

32. The sides of an equlateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area increases when the slide is 10 cm.

33. Find the radius of a sphere if the rate of increasing of its volume is twice what of the the surface area.

34. Write the set of values of x for which the function f(x) = sin x - x is increasing.

35. Mention the values of x for which the function $f(x)=x^3-12x$ is decreasing.

36. Find the interval of x is which the function $y=\frac{\text{lin}\,x}{x}$, x > 0 is increasing.

37. Show that the sum of intercepts on the coordinate axes of any tangent to the curve $\sqrt{x}+\sqrt{y}=\sqrt{a}$ is constant.

38. Show that $2\sin x+3\tan x>3x$ for all $x\in\left(0,\frac{\theta}{2}\right)$.

39. Find the interval in which the function $f(x)=3x^4-4x^3-12x^2+5$ is
   (i) Strictly increasing
   (ii) Strictly decreasing

40. Find the equation of the normal to the curve $x=a\sin^3\theta$, $y=a\cos^3\theta$ at $\theta=\frac{\pi}{4}$.

41. Find the maximum and minimum value of $x+\frac{1}{x}$.

42. For which value of x, the function $f(x)=4-x-x^2$ maximum and minimum.

43. Find the value of x for which the function $f(x)=x^4-4x^3+4x^2-1$ is maximum or minimum.

44. Find the extreme point of the function $f(x)=\sin x\cos x, x\in\left(\frac{\pi}{8},\frac{\pi}{2}\right)$.

45. Show that for all rectangles with a given perimeter, the square has the largest area.

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5. Integrals, Application of Integrals, Integrals, Differential Equation

Each questions carries 4 marks

1. If and $f'(x)=e^x+\frac{1}{1+x^2}$ and $f(0)=1$ then find f(x).
2. Evaluate $\int(\log x)^2dx$
3. Evaluate $\int\frac{2x+9}{(x+3)^2}dx$
4. Evaluate $\int_0^1\frac{x^5(4-x^2)}{\sqrt{1-x^2}}dx$
5. Evaluate $\int\frac{\sin x\cos x}{\sin^2x-2\sin x+3}dx$
6. Evaluate $\int_0^1x^7\sqrt{\frac{1+x^2}{1-x^2}}dx$
7. Evaluate $\int x^2\tan^{-1}x\,dx$
8. Evaluate $\int\frac{1}{x\ln x\sqrt{(\ln x)^2-4}}dx$
9. Evaluate $\int\frac{dx}{(1+x)\sqrt{1-x^2}}$
10. Show that $\int_0^1\frac{\ln x}{\sqrt{1-x^2}}dx=-\frac{\pi}{2}\ln 2$
11. Prove that $\int_0^{\pi/2}\frac{\sin^n x}{\sin^n x+\cos^n x}dx=\frac{\pi}{4}$
12. Integrate $\int\sec x\bullet\tan x\sqrt{\tan^2x-3}\,dx$
13. Evaluate $\int_0^{\pi/4}\frac{1}{\cos x(\cos x+\sin x)}dx$
14. Evaluate $\int_0^1(\tan^{-1}x+\frac{x}{1+x^2})\,dx$
15. Evaluate $\int_0^1[3x]\,dx$

16. Evaluate $\int_0^4 |8-3x|\,dx$

17. Evaluate $\int_0^4 \{[x]+|x|\}\,dx$

18. Evaluate $\int_0^{1.5} [x^2]\,dx$

19. Integrate $\int \frac{xe^x}{(1+x)^2}\,dx$

20. Integrate $\int \frac{a}{b+ce^x}\,dx$

21. Integrate $\int \frac{e^{x-1}+x^{e-1}}{e^x}\,dx$

22. Find the are of the circle $x^2+y^2=2ax$.

23. Find the area bounded by the curve $y^2=x$ and the straight line x = 0, y = 1.

24. Find the area bounded by y = sin x, y = 0 and x = 0, $x=\frac{\pi}{2}$.

25. Find the area bounded by the line y = 2x, x-axis and the ordinate x = 3.

26. Find the area bounded by the curve y = sin x, x-axis from x = 0 to $x=\pi$.

27. Find the area bounded by the curve $y=3x^2+5$, y = 0 and two ordinates x=1 & x=2.

28. Find the area enclosed by $y^2=4ax$ and $x^2=4ay$.

29. Find the area bounded by the curve y = e^x, y=0, x=2 and x=4.

30. Find the area of the trapezium bounded by the sides y = x, x = 0, y = 3 and y = 4.

31. Write the integrating factor of the differential equation $(x-\ln y)\frac{dy}{dx}=-y\ln x$.

32. Solve $\frac{dy}{dx}=\frac{y^2}{xy-x^2}$.

33. Find the particular solution of the following differential equation $\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$ given that $y=\sqrt{3}$ when x = 1.

34. Solve $(x+2y^3)\frac{dy}{dx}=y$.

35. Solve cosec x . $\frac{d^2y}{dx^2}=x$.

36. Solve $y\,dy+e^{-y}x\sin x\,dx=0$.

37. Solve $(x^2-1)\frac{dy}{dx}+2y=1$.

38. Solve $x^2(y-1)dx+y^2(x-1)dy=0$.

39. Find the particular solution of the differential equation $\frac{d^2y}{dx^2}=6x$ given that $y=1$ and $\frac{dy}{dx}=2$ when $x=0$.

40. Find the integrating factor of the differential equation $(1+y^2)dx+(x-e^{-\tan-1y})dy=0$.

41. Solve $\frac{dy}{dx}=\frac{1}{x^2-7x+12}$.

42. Form the differential equation whose general solution is $y=at+be^t$.

43. Find the differential equation whose general solution is $ax^2+by=1$ where a and b are arbitrary constants.

44. Find the differential equation representing family of curves given by $(x-a)^2+2y^2=a^2$ where a is an arbitrary constant.

45. Obtain the different equation whose primitive is $y=Ae^{2x}+Be^{-2x}$.

***

6. Vectors, Three dimensional Geometry

Each questions carries 4 marks

1. If the sum of two unit vectors is a unit vector, then find the magnitude of their difference.

2. ABCD is a parallelogram. Using vector method, prove that the line joining A and the mid point $\overline{BC}$ intersect the diagonal $\overline{BD}$ in the ratio 1:2.

3. Prove that the following vectors can never be coplanar for any real value of λ.

   $(\lambda+1)\hat{i}+2\hat{j}+\hat{k},\ -\hat{i}+\lambda\hat{j}+\hat{k},\ \lambda\hat{i}+\hat{j}+3\hat{k}$.

4. If $\vec{a}=(2,-2,1), \vec{b}=(2,3,6)$ and $\vec{c}=(-1,0,2)$, then find the magnitude and direction of $\vec{a}+\vec{b}-\vec{c}$.

5. Using vector method, prove that an angle inscribed in a semi circle is a right angle.

6. If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular, then prove that $[\vec{a}.(\vec{b}\times\vec{c})]=a^2b^2c^2$.

7. Prove that for any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$,

   $[\vec{a}+\vec{b}\ \ \vec{b}+\vec{c}\ \ \vec{c}+\vec{a}]=2[\vec{a}\ \vec{b}\ \vec{c}]$.

8. If $\hat{i}+\hat{j}+\hat{k}$ and $2\hat{i}-\alpha\hat{j}+3\hat{k}$ are orthogonal to each other then find α.

9. Find $[\vec{a}\ \vec{b}\ \vec{c}]$ when

   $\vec{a}=\hat{i}-2\hat{j}+3\hat{k},\ \vec{b}=2\hat{i}+\hat{j}-\hat{k},\ \vec{c}=\hat{j}+\hat{k},$

10. Find a vector $\vec{b}$ such that $\vec{a}\times\vec{b}=\vec{c}$ and $\vec{a}.\vec{b}=3$ where $\vec{a}=\hat{i}+\hat{j}+\hat{k}$, $\vec{c}=\hat{j}-\hat{k}$.

11. Prove by vector method that in a triangle ABC $c^2=a^2+b^2-2ab\cos C$.

12. Determine the area of the parallelogram whose sides are the vectors $2\hat{i}+2\hat{j}$ and $\hat{i}-\hat{k}$.

13. If the position vectors of the points A, B, C are $2\hat{i}+\hat{j}-\hat{k}$, $3\hat{i}-2\hat{j}+\hat{k}$, $\hat{i}+4\hat{j}-3\hat{k}$ respectively, then prove that A, B, C are collinear.

14. Find the scalar projection of the vector

    $\vec{a}=3\hat{i}+6\hat{j}+9\hat{k}$ on $\vec{b}=2\hat{i}+2\hat{j}-\hat{k}$.

15. Find the value of λ such that the following vectors are coplanar $-\hat{i}+\lambda\hat{j}-\lambda\hat{k}$, $2\hat{i}+4\hat{j}+5\hat{k}$, $-2\hat{i}+4\hat{j}-4\hat{k}$.

16. Vectors $\vec{a}, \vec{b}$ and $\vec{c}$ such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3, |\vec{b}|=5$ and $|\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$.

17. If the magnitude the difference of two unit vectors is $\sqrt{3}$, then find the magnitude of their sum.

18. Let $\vec{a}=\hat{i}+\hat{j}+\hat{k},\ \vec{b}=4\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c}=\hat{i}-2\hat{j}+\hat{k}$. Find a vector of magnitude 6 units which is parallel to the vector $2\vec{a}-\vec{b}+3\vec{c}$.

19. Prove that four points with position vectors $\hat{i}+\hat{j}-3\hat{k},\ 2\hat{i}-\hat{j}-\hat{k},\ -\hat{i}+2\hat{j}-2\hat{k}$ and $2\hat{i}+2\hat{k}$ are coplanar.

20. Write the volume of the parallelopiped whose sides are given by $-\hat{j}, \hat{k}$ and $-\hat{i}$.

21. Find the angle between the lines

    $\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda(3\hat{i}+2\hat{j}+6\hat{k})$ and

    $\vec{r}=7\hat{i}-6\hat{j}+6\hat{k}+\mu(\hat{i}+2\hat{j}+2\hat{k})$.

22. Find the shortest distance between the lines $\frac{x+1}{7}=\frac{y+1}{-6}=\frac{z+1}{1}$ and $\frac{x-3}{1}=\frac{y-5}{-2}=\frac{z-3}{1}$.

23. Find the equation of the plane passing through the point (-1, 3, 2) and perpendicular to the planes x + 2y + 2z = 5 and 3x + 3y + 2z = 5.

24. Prove that the lines $\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2}$ and $3x-2y+z+5=0=2x+3y+4z-4$ are not coplanar.

25. Find the equation of the plane passing through the points (1, 2, -3), (2, 3, -4) and perpendicular to the plane x + y + z + 1 = 0.

26. Find the perpendicular distance of the point (-1, 3, 9) from the line $\frac{x-13}{5}=\frac{y+8}{-8}=\frac{z-31}{1}$.

27. Prove that the measure of the angle between two main diagonals of a cube is $\cos^{-1}\frac{1}{3}$.

28. If the point (1, y, z) lies on the straight line through (3, 2, -1) and (-4, 6, 3), then find y and z.

29. Find the equation of the plane passing through the intersection of the planes x+2y+3z-4=0 and 2x+y-z+5=0 and also perpendicular to the plane 2x-y+2z-3=0.

30. Find the equation of the plane passing through the foot of the perpendicular drawn from the point (a, b, c) on the coordinate planies.

31. Find the equation of the plane passing through the points (2, 1, -1) and (-1, 3, 4) and perpendicular to the plane x- 2y + 4z = 10.

32. Find the value of r if the line $\frac{x-1}{1}=\frac{y+2}{3}=\frac{z-1}{-1}=r$ interst the plane $2x+y+z=9$.

33. Find the ratio in which the line through (1, 3, -1) and (2, 6, -2) is divided by zx - plane.

34. Find the coordinates of the point at which the perpendicular from the origin meets the line joining the points (-9, 4, 5) and (11, 0, -1).

35. Find the image of the point (2, -1, 3) in the plane 3x - 2y + z - 9 = 0.

36. Find the point where the line $\frac{x-2}{1}=\frac{y}{-1}=\frac{z-1}{2}$ meets the plane $2x+y+z=2$.

37. Find the angle between the plane $3x+3z-5=0$ and the line $\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{0}$.

38. Find the equation of the plane which pases through (1, 1, 2) and parallel to the plane $x+2y-z=5$.

39. Find the equation of the plane bisecting the line segment joining (-1, 4, 3) and (5, -2, -1) at right angles.

40. Find the distance between the parallel planes $3x-2y+6z-7=0$ and $3x-2y+6z-14=0$.

41. Find the coordinates of the point of intersection of the line $3x-3=y+2=3-3z$ and the plane $2x+y+z=9$.

42. Find the equation of the plane passing through the points (-2, 3, 5), (7, -7, -5) and (-2, 5, -3).

43. Find the equation of the plane passing through the intersection of the planes $3x+y-z=2$ and $x-y+2z=1$ and the point (1, 0, 2).

44. Find the point of intersection of the line passing through the point (1, 3, -2) and (3, 4, 1) with the plane $x-2y+4z=11$.

45. Find the symmetric form of the equation of the line $x+2y+z-3=0=6x+8y+3z-10$.

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GROUP - B

ANSWERS

2. Relation and Functions, Inverse Trigonometric functions, Linear Programming

1. The given set is Z, the set of integers

   Z = {....–3, –2, –1, 0, 1, 2, 3,..........}

   Given relation R is

   R = {(x,y) : x, y $\in$ Z and x – y is an integer}.

   Reflexive

   For all $x \in Z$, x – x = 0

   $\Rightarrow$ x – x is an integer

   $\Rightarrow$ $(x, y) \in R$

   $\therefore$ The relation R is reflective.

   Symmetric

   For $x, y \in R$

   $x, y \in R$ $\Rightarrow$ x - y is an integer

   $\Rightarrow$ x -y = k where k is an integer

   $\Rightarrow$ y - x = – k

   $\Rightarrow$ y – x is an integer

   $\Rightarrow$ $(x, y) \in R$.

   So the relation R is symmetric.

   Transitive

   For $x, y, z \in Z$

   $(x, y) \in R$ and $(y, z) \in R$

   $\Rightarrow$ x - y is an integer and y – z is an integer

   $\Rightarrow$ x - y = k and y – z = k_1

   where k, k, are integers.

   $\Rightarrow$ (x - y) + (y - z) = k + k_1

   $\Rightarrow$ x – z = k + k_1 which is an integer

   $\Rightarrow$ $(x, z) \in R$

   So the relation R is transitive.

2. Let A be the given set.

   Given that R ans S are two equivalence relation on A.

   $\Rightarrow$ R ans S are reflexive, symmetric and transitive

   (i) Since R and S are reflexive

   $\Rightarrow$ For all $x \in A$, (x, x) $\in$ R and (x, x)$\in$S

   $\Rightarrow$ $(x, x) \in R \cap S$

   $\Rightarrow$ $R \cap S$ is reflexive on the set A.

   (ii) For some $x, y \in A$

   $(x, y) \in R \cap S$

   $\Rightarrow$ $(x, y) \in R$ and $(x, y) \in S$

   $\Rightarrow$ $(y, x) \in R$ and $(y, x) \in S$

   ($\because$ R & S are symmetric)

   $\Rightarrow$ $(y, x) \in R \cap S$

   $\Rightarrow$ $R \cap S$ is symmetric.

   (iii) For some $x, y, z \in A$,

   $(x, y), (y, z) \cap R \cap S$

   $\Rightarrow$ $(x, y), (y, z) \in R$ and $(x, y), (y, z) \in S$

   $\Rightarrow$ $(x, z) \in R$ and $(x, z) \in S$

   $\Rightarrow$ $(x, z) \in R \cap S$

   $\Rightarrow$ $R \cap S$ is transitive.

   Since $R \cap S$ is reflexive, symmetric and transitive, it is an equivalence relation.

3. Given function is $f : R \to R$ and is defined by

   $$f(x) = \frac{2x^2}{x^2 + 1}$$

   For $x_1, x_2 \in R$, $f(x_1) = f(x_2)$

   $$\Rightarrow \frac{2{x_1}^2}{{x_1}^2 + 1} = \frac{2{x_2}^2}{{x_2}^2 + 1}$$

   $$\Rightarrow x_1^2\left(x_2^2 + 1\right) = x_2^2\left(x_1^2 + 1\right)$$

$\Rightarrow\ x_1^2 = x_2^2$

$\Rightarrow\ x_1 = \pm x_2$

$\Rightarrow$ f is not one-one.

Again for x = 1 and –1, use set the same value of f(x).

$\Rightarrow$ The function f(x) is not on to.

4. Given function is $f:R\to R$ such that $f(x)=4x^3+7$.

one-one :

For all $x_1,\ x_2 \in R$

$f(x_1) = f(x_2)$

$\Rightarrow\ 4x_1^3+7=4x_2^3+7$

$\Rightarrow\ x_1^3 = x_2^3$

$\Rightarrow\ x_1 = x_2 \quad (\because\ x_1, x_2 \in R)$

Thus, $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$

$\Rightarrow$ The function is one-one function.

on-to :

To show that f is an on to function, we have to show that Range of f(x) = co domain of f(x).

Given that co domain of f(x) is R.

Let $y = 4x^3 + 7$

$\Rightarrow\ 4x^3 = y-7$

$\Rightarrow\ x^3 = \frac{y-7}{4}$

$\Rightarrow\ x^3 = \left(\frac{y-7}{4}\right)^{\frac{1}{3}}$

From the above we see that for every $y\in R$, we get $x\in R$.

$\Rightarrow$ Range is f(x) = R

$\Rightarrow$ Range is f(x) = co domain of f(x).

$\Rightarrow$ f is an on to function.

Since is one-one and on to, it is a bijective function.

5. Given that $f(x)=\sqrt{x}$ and $g(x)=1-x^2$

$\therefore$ (fog) (x) = f[g(x)]=f$(1-x^2)$=$\sqrt{1-x^2}$

(gof) (x) = g[f(x)]=g $(\sqrt{x})$

$= 1-\left(\sqrt{x}\right)^2 = 1-x$

The natural domain of fog is $-1\le x\le 1$

Natural domain of gof is R which is the the set of real numbers.

6. (i) Let h(x) = f(x) + f(-x)

$\therefore$ h(-x) = f(-x) + f[-(-x)]

= f(-x) + f(x)

= f(x)+f(-x)=h(x)

Thus the function h(x) = f(x) + f(-x) is an even function.

(ii) Let h(x) = f(x) – f(-x)

$\therefore$ h(-x) = f(-x) – f(x)

= –[f(x) – f(-x)]

= – h(x)

So h(x) is an odd function.

7. Let f(x) = cosx.

Let $A=\left\{0,\frac{\pi}{2}\right\},\ B=\left\{\frac{\pi}{2},2\pi\right\}$

$\therefore\ f(A)=\left\{\cos 0,\ \cos\frac{\pi}{2}\right\}$

= { 1, 0 } = {0, 1}

$f(B)=\left\{\cos\frac{\pi}{2},\ \cos 2\pi\right\}$

$= \{0, 1\}$

$\therefore\ f(A)\cap f(B)=\{0, 1\}$ ....(1)

$A\cap B=\left\{\frac{\pi}{2}\right\}$

$\therefore\ f(A\cap B)=\cos\frac{\pi}{2}=\{0\}$

$\therefore\ f(A\cap B)\ne f(A)\cap f(B)$

8. Given that $f : x \to y$ is injective.

Let $x \in A \Leftrightarrow f(x) \in f(A)$ ($\because$ f is injective)

$\Leftrightarrow x \in f^{-1}\left[f(A)\right]$

$\therefore \quad A = f^{-1}\left[f(A)\right]$

9. Given that $f : x \to y$ is injective.

Let A and B are subsets of X.

Let $f(x) \in f(A \cap B)$

$\Leftrightarrow x \in A \cap B$

$\Leftrightarrow x \in A$ and $x \in B$

$\not\Leftrightarrow f(x) \in f(A)$ and $f(x) \in f(B)$

$\not\Leftrightarrow f(x) \in f(A) \cap f(B)$

$\therefore \quad f(A \cap B) = f(A) \cap f(B)$

Conversely suppose that

$f(A \cap B) = f(A) \cap f(B)$

Let f is not injective.

$f(x) \in f(A \cap B) \Leftrightarrow x \in A \cap B$

$\Leftrightarrow x \in A$ and $x \in B$

$\not\Leftrightarrow f(x) \in f(A) \wedge f(x) \in f(B)$

$\not\Leftrightarrow f(x) \in f(A) \cap f(B)$

$\therefore \quad f(A \cap B) = f(A) \cap f(B)$ is false.

So f must be injective.

10. f(x) = sin x, g(x) = x^5

(fog) (x) = f [g(x)] = f(x^5) = sin x^5

(gof) (x) = g [f(x)] = g(sin x) = $\sin^5 x$

$\therefore$ fog $\neq$ gof

11. The given set is $\mathbb{R}$, the set of real numbers. The relation R on the set $\mathbb{R}$ is defined as for $a, b \in \mathbb{R}$, $R = \{(a,b) : a \leq b^2\}$

Reflexive : For all $a \in \mathbb{R}$, $a \leq a^2$ is not true

$$\left[\text{i.e } \frac{1}{2} \leq \left(\frac{1}{2}\right)^2 \text{ is not true}\right]$$

$\Rightarrow$ (a, a) $\in$ R is not true.

So the relation R is reflexive.

Symmetric For $a, b \in \mathbb{R}$, $(a,b) \in \mathbb{R}$

$\Rightarrow a \leq b^2$

$\not\Rightarrow b \leq a^2$

$\not\Rightarrow (b,a) \in R$

$\therefore (a,b) \in R \not\Rightarrow (b,a) \in R$

So the relation R is not symmetric.

Transitive: Here we observe that

for 2, –4, 1 $\in \mathbb{R}$,

$2 \leq (-4)^2, -4 \leq 1$

But $2 \not\leq 1$.

Thus, $(2,4) \in R$, $(-4,1) \in R$

But (2, 1) $\notin$ R .

So the relation R is not transitive.

12. Given that

$$f(m,n) = \begin{cases} 5 & \text{if } m < n \\ f(m-n, n+2) + m & \text{if } m \geq n \end{cases}$$

f(5,3) = f(5-3, 3+2) + 5
= f (2, 5) + 5
= 5 + 5 = 10.

13. (i) We know addition of two rational numbers is a rational number. Also multiplication of two rational numbers is a rational number.

Here a, b are rational numbers other than 1.

So a + b – ab is a rational number.

$\Rightarrow$ * is a binary operation on the set S.

(ii) **Commutative** For $a, b \in S$

a * b = a + b – ab
= b + a – ba
= b + a

Here * is commutative.

Associative For a, b, c $\in$ S

$\therefore$ (a * b)*c = (a+b–ab)*c
= (a+b–ab)*c
= a+b-ab+c-(a+b-ab)c
= a+b+c-ab-bc-ac+abc .....(1)

a*(b*c) = a*(b+c-bc)

= a+b+c-bc-a(b+c-bc)

= a+b+c-ab-bc-ca+abc .....(2)

$\therefore$ (a*b)*c = a*b(b*c).

$\Rightarrow$ * is associative.

14. Given binary operation is

a * b = min {a, b} and is defined on the set {1,2,3,4,5}

The operation table for * is given as follows :

*	1	2	3	4	5
1	1	1	1	1	1
2	1	2	2	2	2
3	1	2	3	3	3
4	1	2	3	4	5
5	1	2	3	4	5

15. Given that a*b = L.C.M. of a and b

L.C.M of 20 and 16 =80

$\Rightarrow$ 20 * 16 = 80

(i) **Commutative** For a, b $\in$ N

a*b = L.C.M. of a and b

b*a = L.C.M of b and a

= L.C.M. of a and b

= a*b

Thus * is commutative

(ii) For a, b, c $\in$ N,

a * (b*c) = a* (L.C.M. of and c)

= L.C.M. of a, b, c ....(1)

Again (a*b)*c = (L.C.M of a and b)*c

= L.C.M. of a, b, c ....(2)

From (1) and (2), we have

a*(b*c) = (a*b)*c.

16. We know $\sin^{-1}x = \tan^{-1}\dfrac{x}{\sqrt{1-x^2}}$

and $2\tan^{-1}x = \tan^{-1}\dfrac{2x}{1-x^2}$

L.H.S = $\sin^{-1}\dfrac{4}{5} + 2\tan^{-1}\dfrac{1}{3}$

$$= \tan^{-1}\frac{4/5}{\sqrt{1-\frac{16}{25}}} + \tan^{-1}\frac{2.\frac{1}{3}}{1-\frac{1}{9}}$$

$$= \tan^{-1}\frac{4}{3} + \tan^{-1}\frac{3}{4}$$

$$= \tan^{-1}\frac{4}{3} + \cot^{-1}\frac{4}{3} = \frac{\pi}{2}$$

17. We shall show that

$$2\left(\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{2}{9}\right) = \tan^{-1}\frac{4}{3}$$

LHS $= 2\left(\tan^{-1}\dfrac{1}{4} + \tan^{-1}\dfrac{2}{9}\right)$

$$= 2\tan^{-1}\left(\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}.\frac{2}{9}}\right)$$

$$= 2\tan^{-1}\left(\frac{17}{34}\right) = 2\tan^{-1}\frac{1}{2}$$

$$= \tan^{-1}\frac{2.\frac{1}{2}}{1-\frac{1}{4}} = \tan^{-1}\frac{4}{3}$$

18. L.H.S $= \tan^{-1}\dfrac{3}{4} + \tan^{-1}\dfrac{3}{5} - \tan^{-1}\dfrac{8}{19}$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{3}{5}}{1-\frac{3}{4}.\frac{3}{5}}\right) - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\frac{27}{11} - \tan^{-1}\frac{8}{19}$$

$$= \tan^{-1}\left(\frac{\frac{27}{11}-\frac{8}{19}}{1+\frac{27}{11}.\frac{8}{19}}\right) = \tan^{-1}1 = \frac{\pi}{4}$$

19. Given that $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$

$\Rightarrow \cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z$

$\Rightarrow \cos^{-1}\left[xy - \sqrt{(1-x^2)(1-y^2)}\right] = \pi - \cos^{-1}z$

$\Rightarrow xy - \sqrt{(1-x^2)(1-y^2)} = \cos(\pi - \cos^{-1}z)$

$= -\cos\cos^{-1}z$

$= -z$

$\Rightarrow xy + z = \sqrt{(1-x^2)(1-y^2)}$

$\Rightarrow (xy+z)^2 = (1-x^2)(1-y^2)$

$\Rightarrow x^2 + y^2 + z^2 + 2xyz = 1$

20. Given that $\sin^{-1}x + \sin^{-1}y + \sin^{-1}z = \pi$

Let $\sin^{-1}x = A$, $\sin^{-1}y = B$, $\sin^{-1}z = C$

$\Rightarrow x = \sin A$, $y = \sin B$, $z = \sin C$

$\Rightarrow \cos A = \sqrt{1-\sin^2 A} = \sqrt{1-x^2}$

$\cos B = \sqrt{1-y^2}$, $\cos C = \sqrt{1-z^2}$

L.H.S = $x\sqrt{1-x^2} + y\sqrt{1-y^2} + z\sqrt{1-z^2}$

$= \sin A\cos A + \sin B\cos B + \sin C\cos C$

$= \frac{1}{2}(\sin 2A + \sin 2B + \sin 2C)$

$= \frac{1}{2}.4\sin A\sin B\sin C$

$= 2xyz.$

21. Given that $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \pi$

$\Rightarrow \tan^{-1}x + \tan^{-1}y = \pi - \tan^{-1}z$

$\Rightarrow \tan^{-1}\frac{x+y}{1-xy} = \pi - \tan^{-1}z$

$\Rightarrow \frac{x+y}{1-xy} = \tan(\pi - \tan^{-1}z)$

$= -\tan\tan^{-1}z$

$= -z$

$\Rightarrow x + y = -z(1-xy)$

$\Rightarrow x + y + z = xyz$

22. L.H.S $= \tan\left(2\tan^{-1}\frac{1}{5} - \frac{\pi}{4}\right)$

$= \tan\left(\tan^{-1}\frac{2.\frac{1}{5}}{1-\frac{1}{25}} - \tan^{-1}1\right)$

$= \tan\left(\tan^{-1}\frac{5}{12} - \tan^{-1}1\right)$

$= \tan\tan^{-1}\left(\frac{\frac{5}{12}-1}{1+\frac{5}{12}.1}\right)$

$= -\frac{7}{12}$

23. Given that $u = \cot^{-1}\left(\sqrt{\cos\alpha}\right) - \tan^{-1}\left(\sqrt{\cos\alpha}\right)$

$\Rightarrow u = \tan^{-1}\left(\frac{1}{\sqrt{\cos\alpha}}\right) - \tan^{-1}\sqrt{\cos\alpha}$

$\Rightarrow u = \tan^{-1}\left(\frac{\frac{1}{\sqrt{\cos\alpha}} - \sqrt{\cos\alpha}}{1+\frac{1}{\sqrt{\cos\alpha}}\sqrt{\cos\alpha}}\right)$

$\Rightarrow \tan u = \frac{1-\cos\alpha}{2\sqrt{\cos\alpha}}$

$\Rightarrow 1+\tan^2 u = 1+\frac{(1-\cos\alpha)^2}{4\cos\alpha}$

$\Rightarrow \sec^2 u = \frac{(1-\cos\alpha)^2}{4\cos\alpha}$

$\Rightarrow \sec u = \frac{1+\cos\alpha}{2\sqrt{\cos\alpha}}$

$\sin u = \frac{\tan u}{\sec u} = \frac{1-\cos\alpha}{1+\cos\alpha} = \tan^2\frac{\alpha}{2}$

24. Let $\sin^{-1}\sqrt{\frac{x-q}{p-q}}=\theta$

$\Rightarrow \sin\theta=\sqrt{\frac{x-q}{p-q}}$

$\cos\theta = \sqrt{1-\sin^2\theta}=\sqrt{1-\frac{x-q}{p-q}}=\sqrt{\frac{p-x}{p-q}}$

$\Rightarrow \theta=\cos^{-1}\sqrt{\frac{p-x}{p-q}}$

$\Rightarrow \sin^{-1}\sqrt{\frac{x-q}{p=q}}=\cos^{-1}\sqrt{\frac{p-x}{p-q}}$

$\tan\theta=\frac{\sin\theta}{\cos\theta}=\sqrt{\frac{x-q}{p-x}}$

$\Rightarrow \theta=\tan^{-1}\sqrt{\frac{x-q}{p-x}}$

$\therefore \sin^{-1}\sqrt{\frac{x-q}{p-q}}=\cos^{-1}\sqrt{\frac{p-x}{p-q}}=\tan^{-1}\sqrt{\frac{x-q}{p-x}}$

25. L.H.S $= \tan^{-1}x + \cot^{-1}(x + 1)$

$= \tan^{-1}x+\tan^{-1}\frac{1}{x+1}$

$= \tan^{-1}\left(\frac{x+\frac{1}{x+1}}{1-x\frac{1}{x+1}}\right)$

$= \tan^{-1}(x^2 + x + 1)$

26. L.H.S. $= 4\left(\cot^{-1}\frac{3}{2}+\text{cosec}^{-1}\sqrt{26}\right)$

$= 4\left(\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{5}\right)$

$= 4\tan^{-1}\left(\frac{\frac{2}{3}+\frac{1}{5}}{1-\frac{2}{3}.\frac{1}{5}}\right)=4\tan^{-1}1=\pi$

$= \tan^{-1}(x^2+x+1)$

26. L.H.S $= 4\left(\cot^{-1}\frac{3}{2}+\text{cosec}^{-1}\sqrt{26}\right)$

$= 4\left(\tan^{-1}\frac{2}{3}+\tan^{-1}\frac{1}{5}\right)$

$= 4\tan^{-1}\left(\frac{\frac{2}{3}+\frac{1}{5}}{1-\frac{2}{3}.\frac{1}{5}}\right)=4\tan^{-1}1=\pi$

27. Given that $\angle A=90^0$

$\Rightarrow b^2 + c^2 = a^2$

$\therefore \tan^{-1}\frac{b}{a+c}+\tan^{-1}\frac{c}{a+b}$

$= \tan^{-1}\left[\frac{\frac{b}{a+c}+\frac{c}{a+b}}{1-\frac{b}{a+c}.\frac{c}{a+b}}\right]$

$= \tan^{-1}\left(\frac{ab+ac+a^2}{a^2+ac+ab}\right)$

$= \tan^{-1}1=\frac{\pi}{4}.$

28. L.H.S $=\tan^{-1}x+\tan^{-1}\left(\frac{2x}{1-x^2}\right)$

$=\tan^{-1}\left(\frac{x+\frac{2x}{1-x^2}}{1-x.\frac{2x}{1-x^2}}\right)$

$=\tan^{-1}\left(\frac{x\left(1-x^2\right)+2x}{1-x^2-2x^2}\right)$

$=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$

29. L.H.S $= \cos\left[\tan^{-1}\left\{\sin\left(\cot^{-1}x\right)\right\}\right]$

Let $\cot^{-1}x = \theta \Rightarrow x = \cot\theta$

L.H.S $= \cos\left[\tan^{-1}(\sin\theta)\right]$

$= \cos\left[\tan^{-1}\frac{1}{\operatorname{cosec}\theta}\right]$

$= \cos\left[\tan^{-1}\frac{1}{\sqrt{1+x^2}}\right]$ ....(1)

Let $\tan^{-1}\frac{1}{\sqrt{1+x^2}} = \phi$

$\Rightarrow \tan\phi = \frac{1}{\sqrt{1+x^2}}$

From (1), L.H.S

$= \cos\phi = \frac{1}{\sec\phi} = \frac{1}{\sqrt{1+\tan^2\phi}}$

$= \frac{1}{\sqrt{1+\frac{1}{1+x^2}}} = \sqrt{\frac{1+x^2}{2+x^2}}$.

30. Let $x = \tan^2\theta \Rightarrow \tan\theta = \sqrt{x}$

$\Rightarrow \theta = \tan^{-1}\sqrt{x}$

R.H.S $= \frac{1}{2}\cos^{-1}\left(\frac{1-x}{1+x}\right)$

$= \frac{1}{2}\cos^{-1}\left(\frac{1-\tan^2\theta}{1+\tan^2\theta}\right)$

$= \frac{1}{2}\cos^{-1}\cos 2\theta$

$= \frac{1}{2}.2\theta = \theta = \tan^{-1}\sqrt{x}$

31. L.H.S $= \sec^2(\tan^{-1}3) - \operatorname{cosec}^2(\cot^{-1}3)$

$= 1 + \tan^2(\tan^{-1}3) - [1+\cot^2\cot^{-1}3]$

$= 1 + 3^2 - (1+3^2) = 0$

32. Given that $\cos\left(\tan^{-1}x\right) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$\Rightarrow \sin\left(\frac{\pi}{2} - \tan^{-1}x\right) = \sin\left(\cot^{-1}\frac{3}{4}\right)$

$\Rightarrow \frac{\pi}{2} - \tan^{-1}x = \cot^{-1}\frac{3}{4}$

$\Rightarrow \tan^{-1}x + \cot^{-1}\frac{3}{4} = \frac{\pi}{2}$

$\Rightarrow x = \frac{3}{4}$

33. $\tan^{-1}\left(\frac{1-x}{1+x}\right) = \frac{1}{2}\tan^{-1}x$

$\Rightarrow 2\tan^{-1}\left(\frac{1-x}{1+x}\right) = \tan^{-1}x$

$\Rightarrow \tan^{-1}\frac{2\left(\frac{1-x}{1+x}\right)}{1-\left(\frac{1-x}{1+x}\right)^2} = \tan^{-1}x$

$\Rightarrow \tan^{-1}\left(\frac{1-x^2}{2x}\right) = \tan^{-1}x$

$\Rightarrow \frac{1-x^2}{2x} = x$

$\Rightarrow 1 - x^2 = 2x^2$

$\Rightarrow 3x^2 = 1$

$\Rightarrow x^2 = \frac{1}{3}$

$\Rightarrow x = \pm\frac{1}{\sqrt{3}}$

34. Given equation is $\tan^{-1}x + 2\cot^{-1}x = \frac{2\pi}{3}$

$$\Rightarrow \tan^{-1}x + 2\tan^{-1}\frac{1}{x} = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}x + \tan^{-1}\left(\frac{2\frac{1}{x}}{1-\frac{1}{x^2}}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}x + \tan^{-1}\left(\frac{2x}{x^2-1}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left[\frac{x+\frac{2x}{x^2-1}}{1-x.\frac{2x}{x^2-1}}\right] = \frac{2\pi}{3}$$

$$\Rightarrow \tan^{-1}\left(\frac{x^3+x}{-1-x^2}\right) = \frac{2\pi}{3}$$

$$\Rightarrow \frac{x^3+x}{-1-x^2} = \tan\frac{2\pi}{3}$$

$$= \tan\left(\pi - \frac{\pi}{3}\right) = -\tan\frac{\pi}{3}$$

$$\Rightarrow \frac{x^3+x}{-1-x^2} = -\sqrt{3} \Rightarrow x = \sqrt{3}$$

35. L.H.S = $\cos^{-1}x + \cos^{-1}\left[\frac{x}{2} + \frac{\sqrt{3-3x^2}}{2}\right]$

Let $\cos^{-1}x = \alpha \Rightarrow x = \cos\alpha$

L.H.S = $\alpha + \cos^{-1}\left[\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sqrt{1-\cos^2\alpha}\right]$

$$= \alpha + \cos^{-1}\left[\frac{1}{2}\cos\alpha + \frac{\sqrt{3}}{2}\sin\alpha\right]$$

$$= \alpha + \cos^{-1}\left[\cos\frac{\pi}{3}\cos\alpha + \sin\frac{\pi}{3}\sin\alpha\right]$$

$$= \alpha + \cos^{-1}\cos\left(\frac{\pi}{3} - \alpha\right)$$

$$= \alpha + \frac{\pi}{3} - \alpha = \frac{\pi}{3}$$

36. Given equation is

$$\tan^{-1}(x+2) + \tan^{-1}(x-2) = \tan^{-1}\frac{8}{79}$$

$$\Rightarrow \tan^{-1}\frac{(x+2)+(x-2)}{1-(x+2)(x-2)} = \tan^{-1}\frac{8}{79}$$

$$\Rightarrow \tan^{-1}\left(\frac{2x}{5-x^2}\right) = \tan^{-1}\frac{8}{79}$$

$$\Rightarrow \frac{2x}{5-x^2} = \frac{8}{79}$$

$$\Rightarrow 4x^2 + 79x - 20 = 0$$

$$\Rightarrow (x + 20)(4x - 1) = 0$$

$$\Rightarrow x = \frac{1}{4} \quad (x = -20 \text{ is rejected})$$

37. Given equation is

$$\tan^{-1}\left(\frac{1+x}{1-x}\right) = \frac{\pi}{4} + \tan^{-1}x$$

$$\Rightarrow \tan^{-1}\left(\frac{1+x}{1-x}\right) - \tan^{-1}x = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{\frac{1+x}{1-x} - x}{1+\frac{1+x}{1-x}.x}\right) = \frac{\pi}{4}$$

$$\Rightarrow \tan^{-1}\left(\frac{1+x^2}{1-x^2}\right) = \frac{\pi}{4}$$

$$\Rightarrow \frac{1+x^2}{1-x^2} = \tan\left(\frac{\pi}{4}\right) = 1$$

Which is true for all x

The solution is {x : 0 < x < 1}.

38. The given equation is

$$\cos(2\sin^{-1}x) = \frac{1}{9} \quad \text{........(1)}$$

Let $\sin^{-1}x = \theta \quad \Rightarrow \quad x = \sin\theta$

Equation (1) becomes

$$\cos 2\theta = \frac{1}{9}$$

$$\Rightarrow \quad 1 - 2\sin^2\theta = \frac{1}{9}$$

$$\Rightarrow \quad 2\sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$$

$$\Rightarrow \quad \sin^2\theta = \frac{4}{9}$$

$$\Rightarrow \quad \sin\theta = \pm\frac{2}{3}$$

$$\Rightarrow \quad x = \pm\frac{2}{3}$$

Since x > 0, $x = \frac{2}{3}$

39. The given inequations are

$2x + y \geq 6$ .... (1)

$x - y \leq 3,$ .... (2)

$x \geq 0,\ y \geq 0$ .... (3)

Changing the inequation to equations, we get

2x + y = 6 .... (1)

x – y = 3 .... (2)

x = 0, y = 0 .... (3)

The line (4) intersect the axes at two points (3, 0) and (0, 6).

Putting (0, 0) is (1), we get

$2.0 + 0 \ \geq \ 6 \ \Rightarrow \ 0 \ \geq \ 6$ which is false.

So half plane is away from the origin.

The line (5) intersect the axis at (3, 0) and (0, -3)

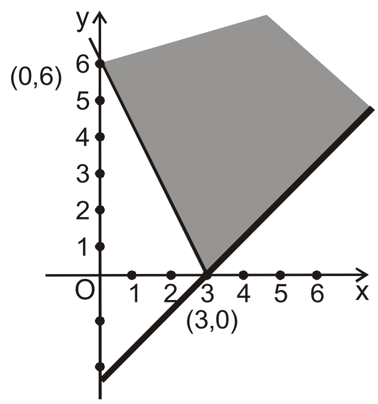
Putting (0, 0) in the inequality (2), we get

$0,0 \leq 3$

$\Rightarrow \ 0 \leq 3$ which is true.

So half plane is towards the origin.

The graphical representation of the lines are given below.



The shaded portion is the given feasible region.

40. The given L.P.P is

Maximize z = 20x + 30y ....(1)

Subject to $3x + 5y \leq 15$, ...(2)

$x, y \geq 0$ ....(3)

Writing the inequations as equations, we get

3x + 5y =15, ...(4)

x=0, y = 0 ....(5)

The line (4) intersect the axes at (5, 0) and (0,3).

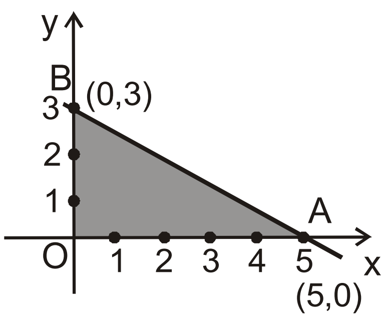
Putting (0, 0) in the inequation (2), we get

$3.0 + 5.0 \leq 15$

$\Rightarrow \ 0 \leq 15$ which is true.

So the half plane is towards the origin.

OAB is the feasible region,



where 0(0,0), A is (5, 0) and B is (0, 3)

The values of z at different pints are as follows

Point	x	y	$z=20x+30y$
0	0	0	$z=0$
A	5	0	$z=20\times x+30.0=100$
B	0	3	$z=20\times 0+30\times 3=90$

Maximum value of z = 100

It is obtained at A where x = 5, y = 0.

41. The given L.P.P. is

Minimize $z = 6x_1 + 7x_2$ ....(1)

Subject to $x_1 + 2x_2 \geq 2$ ....(2)

$x_1, x_2 \geq 0$ ....(3)

Changing the inequations an equations, we get

$x_1 + 2x_2 = 2$ ....(4)

$x_1 = 0,\ x_2 = 0$ ....(5)

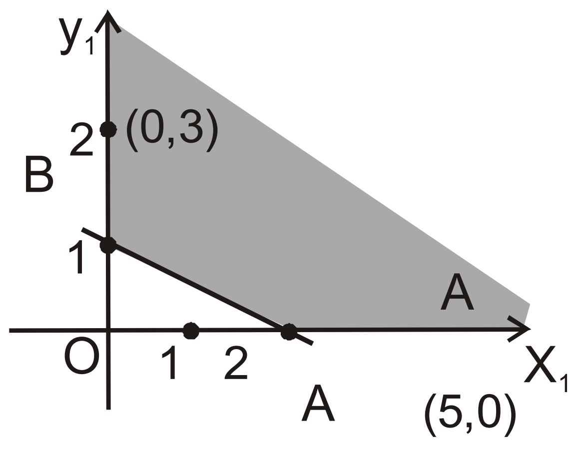
The line (4) intersect the axes at A(2, 0) and B(0,1).

Putting (0, 0) is (2), we get we get

$0 + 2.0 \geq 2$

$\Rightarrow\ 0 \leq 2$ which is false.

So half plane is away from the origin.



The feasible region is

X_1 ABY,

The value of the objective function in the corner points are given below

Point	x_1	x_2	$z = 6x_1 + 7x_2$
A	2	0	$z = 6\times 2 + 7.0 = 12$
B	0	1	$z = 6.0 + 7.1 = 7$

The minimum value of z is 7.

It is obtained at B where $x_1 = 0,\ -x_2=1$.

42. The given L.P.P is

Minimize $z = 3x_1 + 5x_2$ ....(1)

Subject to $5x_1 + 3x_2 \leq 30$ ....(2)

$x_1, +2x_2 \leq 12$ ....(3)

$2x_1 + 5x_2 \leq 20$ ....(4)

and $x_1,\ x_2 \geq 0$ ....(5)

The point (2, 3) satisfies the constraents (2), (3) and (4) and non negative restrictions (5).

So the point (2, 3) is a feasible solutions.

But the point (-3, 4) does not satisfie the constraents and non-negative restriction (5).

So the point (–3, 4) is not a feasible solution.

43. Given L.P.P is

Minimize $z = 5x + 7y$ ....(1)

Subject to $2x + y \geq 8$ ....(2)

$x + 2y \geq 10$ ....(3)

$x \geq 0,\ y \geq 0$ ....(4)

Changing the inequations to equations, we get

$2x + y = 8$ ....(5)

$x + 2y = 10$ ....(6)

$x = 0,\ y = 0$ ....(7)

The line (5) intersects the coordinate axes at (4,0) and (0, 8).

Putting (0,0) (2) we get

$2.0+0 \geq 8 \Rightarrow 0 \geq 8$ which is false.

So half plane is away from the origin.

The line (6) intersects the coordinate axes at (10,0) and (0, 5).

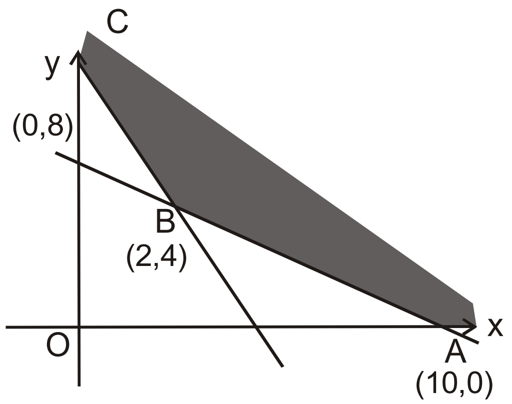
Putting (0,0) equation (3), we get

$0 + 2.0 \geq 10$

$\Rightarrow 0 \geq 101$ which is false.

So half plane is away from the origin.

The shaded portion is the feasible solution.



X ABCY is the feasible region.

Its vertices are A(10,0)

B (2,4) and (0, 8)

The value of the objective function is given in the following table.

Point	x	y	$2=5x+7y$
A	10	0	$z=5\times10+7\times0=50$
B	2	4	$2=5\times2+7\times4=10+28=38$
C	0	8	$z=5.0+7.8=56$

The minimum value of z is 38. It is obtained when x = 2, y=4.

44. The given L.P.P. is

Maximize $z = 5x + 3y$ ....(1)

Subject to $3x+5y \leq 15$ ....(2)

$3x+2y \leq 10$ ....(3)

$x \geq 0,\ y \geq 0$ ....(4)

Changing the inequations to equations, we get

$3x + 5y = 15$ ....(5)

$5x + 2y = 10$ ....(6)

$x = 0,\ y = 0$ ....(7)

The line (5) intersects the coordinate axes at (5,0) and (0, 3).

The line (6) intersects the coordinate axes at (2,0) and (0, 5).

Putting (0,0) inequation (2) and (3), we have $0 \leq 5$ and $0 \leq 10$ wich is true.

So half plane is towards the origin.

The graph of the problem is as shown as the figure.

O ABC is the feasible region whose vertices are O(0, 0), A(2, 0), B$\left(\frac{20}{19},\frac{45}{19}\right)$ and C (0, 3).

The value of z the vertices are given in the following table.

Point	x	y	$z=5x+3y$
O	0	0	$z=0$
A	2	0	$z=5.2+3.0=10$
B	$\frac{20}{19}$	$\frac{45}{19}$	$z=5.\frac{20}{19}+3.\frac{45}{19}=\frac{235}{19}$
C	0	3	$x=5.0+3.3=9$

Maximum value of z is $\frac{235}{19}$.

It is obtained when $x=\frac{20}{19},\ y=\frac{14}{19}$.

45. This problem is same as no 44.

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3. Matrices, Determinants & Probability

1. Given that

$$A = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 3 & -1 \end{bmatrix}$$

$$AB = \begin{bmatrix} 1 & -2 & 2 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 1 & 2 \\ 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.2 + (-2).1 + 2.3 & 1.4 + (-2).2 + 2(-1) \\ 3.2 + 1.1 + (-1).3 & 3.4 + 1.2 + (-1)(-1) \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 2 + 6 & 4 - 4 - 2 \\ 6 + 1 - 3 & 12 + 2 + 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ 4 & 15 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 6 & 4 \\ -2 & 15 \end{bmatrix} \quad .... \quad .... (1)$$

$$B^T = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & -1 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 2 & -1 \end{bmatrix}$$

$$B^T A^T = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -2 & 1 \\ 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 - 2 + 6 & 6 + 1 - 3 \\ 4 - 4 - 2 & 12 + 2 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 4 \\ -2 & 15 \end{bmatrix} .... \quad .... (2)$$

From (1) and (2), we have

$$(AB)^T = B^T A^T$$

2. Given that $A = \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$

We shall show that $A^k = \begin{bmatrix} 1 + 2k & -4k \\ k & 1 - 2k \end{bmatrix} ...(1)$

Let p(k) be the above statement.

First we shall show that P(1) is true.

Taking k=1 in (1), we get

$$A' = \begin{bmatrix} 1 + 2 & -4 \\ 1 & 1 - 2 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 3 & -4 \\ 1 & -2 \end{bmatrix} \text{ which is true.}$$

Let P(m) be true

i.e. $A^m = \begin{bmatrix} 1 + 2m & -4m \\ m & 1 - 2m \end{bmatrix}$

We shall have to show that P(m+1) is true.

L.H.S. of P(M+1) = $P(m+1) = A^{m+1} = A^m.A$

$$= \begin{bmatrix} 1 + 2m & -4m \\ m & 1 - 2m \end{bmatrix} \begin{bmatrix} 3 & -4 \\ 1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} (1 + 2m)3 = 4m & -4(1 + 2m) + (-1)(-4m) \\ 3m + 1 - 2m & -4m + (-1)(1 - 2m) \end{bmatrix}$$

$$= \begin{bmatrix} 3 + 2m & -4 - 4m \\ m + 1 & -2m - 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 2(m+1) & -4(m+1) \\ m + 1 & 1 - 2(m+1) \end{bmatrix} = \text{RHS of P (m+1)}$$

So P (m+1) is true.

Here we see that

(i) P(i) is true

(ii) P(m) is true $\Rightarrow$ P(m+1) is true

So according to the principal of induction, P(k) is true for all $k \in N$.

3. Given that $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x & 2 \\ 1 & y \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -1 & 4 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} x-2 & 2-2y \\ 3x+2 & 6+2y \end{bmatrix} = \begin{bmatrix} -3 & 4 \\ -1 & 4 \end{bmatrix}$

Equating the corresponding term, we get

$x - 2 = -3 \Rightarrow x = -3 + 2 = -1$

$2 - 2y = 4 \Rightarrow 2y = 2 - 4 = -2$

$\Rightarrow y = -1$

4. Given that

$$A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$A^2 = A.A = \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}\begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1+3-5 & -3-9+15 & -5-15+25 \\ -1-3+5 & 3+9-15 & 5-15-25 \\ 1+3-5 & -3-9+15 & -5-15+25 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 3 & 5 \\ 1 & -3 & -5 \\ -1 & 3 & 5 \end{bmatrix} = A.$$

$A^3 = A^2.A = A.A = A^2 \ (\because A^2 = A)$

5. Given that $A = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix}$$

$A^2 - 5A + 7I$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - 5\begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} + 7\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 \\ -5 & 3 \end{bmatrix} - \begin{bmatrix} 15 & 5 \\ -5 & 10 \end{bmatrix} + \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

6. Let $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & 5 \\ 1 & 3 \end{vmatrix} = 6 - 5 = 1$$

$A_{11} = (-1)^{1+1}\ 3 = 3$

$A_{12} = (-1)^{1+2}\ 1 = -1$

$A_{21} = (-1)^{2+1}\ 5 = -5$

$A_{22} = (-1)^{2+2}\ 2 = 2$

Matrix of the cofactors = $= \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$

Adj $A = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = \frac{1}{1}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$$

7. Given that

$$\begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 0 & 5 & 1 \\ 0 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ x \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 1+3+2x \\ 0+5+x \\ 0+3+2x \end{bmatrix} = 0$$

$$\Rightarrow \begin{bmatrix} 1 & x & 1 \end{bmatrix}\begin{bmatrix} 4+2x \\ 5+x \\ 3+2x \end{bmatrix} = 0$$

$\Rightarrow 4 + 2x + x(5 + x) + 1.(3. + 2x) = 0$

$\Rightarrow x^2 + 9x + 7 = 0$

$$\Rightarrow x = \frac{-9 \pm \sqrt{53}}{2}$$

8. Given that

$$2A+B+C=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix} \quad \text{... (1)}$$

$$A+B+C=\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix} \quad \text{... (2)}$$

$$A+B-C=\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix} \quad \text{... (3)}$$

Subtracting (2) from (1), we get

$$A=\begin{bmatrix}1 & 2\\ 3 & 0\end{bmatrix}-\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}$$

Subtracting (3) from (2), we get

$$2C=\begin{bmatrix}0 & 1\\ 2 & 1\end{bmatrix}-\begin{bmatrix}1 & 2\\ 1 & 0\end{bmatrix}=\begin{bmatrix}-1 & -1\\ 1 & 1\end{bmatrix}$$

$$\Rightarrow C=\frac{1}{2}\begin{bmatrix}-1 & -1\\ 2 & 1\end{bmatrix}-\begin{bmatrix}-\frac{1}{2} & -\frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{bmatrix}$$

From (2), we get $B=\begin{bmatrix}-\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{3}{2}\end{bmatrix}$

9. Given that $A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$

$$A^2=A.A=\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}$$

$$=\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}$$

$$A^2-3A+2I=\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}-3\begin{bmatrix}2 & 0 & 1\\ 2 & 1 & 3\\ 1 & -1 & 0\end{bmatrix}+2\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}5 & -1 & 2\\ 9 & -2 & 5\\ 0 & -1 & -2\end{bmatrix}-\begin{bmatrix}6 & 0 & 3\\ 6 & 3 & 9\\ 3 & -3 & 0\end{bmatrix}+\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$$=\begin{bmatrix}1 & -1 & -1\\ 3 & -3 & -4\\ -3 & -4 & 0\end{bmatrix}$$

10. Given that $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$

$$A^2=A.A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$$

$$=\begin{bmatrix}9 & 8 & 8\\ 8 & 9 & 8\\ 8 & 8 & 9\end{bmatrix}$$

$$A^2-4A-5I=\begin{bmatrix}9 & 8 & 8\\ 8 & 9 & 8\\ 8 & 8 & 9\end{bmatrix}-4\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}-5\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$=\begin{bmatrix}9 & 8 & 8\\ 8 & 9 & 8\\ 8 & 8 & 9\end{bmatrix}-\begin{bmatrix}4 & 8 & 8\\ 8 & 4 & 8\\ 8 & 8 & 4\end{bmatrix}-\begin{bmatrix}5 & 0 & 0\\ 0 & 5 & 0\\ 0 & 0 & 5\end{bmatrix}$$

$$=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=0$$

11. Given matrix equation is

$$\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}A=\begin{bmatrix}-4 & 1\\ 7 & 7\end{bmatrix}$$

$$\Rightarrow A=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}^{-1}\begin{bmatrix}-4 & 1\\ 7 & 7\end{bmatrix} \quad \text{... (1)}$$

Let $B=\begin{bmatrix}1 & -1\\ 2 & 3\end{bmatrix}$

$B_{11}=(-1)^{1+1}.3=3$

$B_{12} = (-1)^{1+2}.2 = -2$

$B_{21} = (-1)^{2+1}(-1) = 1$

$B_{22} = (-1)^{2+2}.2 = 2$

Matrix of the cofactors $= \begin{bmatrix} 3 & -2 \\ 1 & 2 \end{bmatrix}$

$\text{Adj } B = \begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$

$|B| = \begin{vmatrix} 1 & -1 \\ 2 & 3 \end{vmatrix} = 3 - (-2) = 5$

$B^{-1} = \frac{1}{|B|}(\text{adj } B) = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}$

From (1), we get

$A = \frac{1}{5}\begin{bmatrix} 3 & 1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} -4 & 1 \\ 7 & 7 \end{bmatrix}$

$= \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}.$

12. Given matrix is $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$

$A_{11} = \text{cofactor of } 1 = (-1)^{1+1}\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 1 - 4 = -3$

$A_{12} = 2,\ A_{13} = -1,\ A_{21} = 3,\ A_{22} = -1,\ A_{23} = -1$

$A_{31} = 0,\ A_{32} = -2,\ A_{33} = 1$

Matrix of the cofactors $= \begin{bmatrix} -3 & 2 & -1 \\ 3 & -1 & -1 \\ 0 & -2 & 1 \end{bmatrix}$

$\text{Adj } A = \begin{bmatrix} -3 & 3 & 0 \\ 2 & -1 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

13. Let A be the required matrix.

According to the question,

$A + \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix}$

$\Rightarrow A = \begin{bmatrix} 4 & -1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -3 \\ -4 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 7 & -5 \end{bmatrix}$

14. Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}$

We known A = IA

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$\Rightarrow \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A \ (R_2 \to R_2 - 2R_1)$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix} A \ (R_1 \to R_1 - 3R_2)$

$\Rightarrow A^{-1} = \begin{bmatrix} 7 & -3 \\ -2 & 1 \end{bmatrix}.$

15. Let $A = \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix}$

$\therefore$ A = IA

$\Rightarrow \begin{bmatrix} 4 & 5 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A$

$\Rightarrow \begin{bmatrix} 1 & \frac{5}{4} \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & 1 \end{bmatrix} A \quad (R_1 \to \frac{1}{4}R_1)$

$\Rightarrow \begin{bmatrix} 1 & \frac{5}{4} \\ 0 & \frac{1}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ -\frac{3}{4} & 1 \end{bmatrix} A \quad (R_2 \to R_2 - 3R_1)$

$\Rightarrow \begin{bmatrix} 1 & \frac{5}{4} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & 0 \\ -3 & 4 \end{bmatrix} A \quad (R_2 \to 4R_2)$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix} A \quad (R_1 \to R_1 - \frac{5}{4}R_2)$

$\Rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = A^{-1} A$

$\therefore A^{-1} = \begin{bmatrix} 4 & -5 \\ -3 & 4 \end{bmatrix}$

16. L.H.S. $= \begin{vmatrix} bc & a & a^2 \\ ca & b & b^2 \\ ab & c & c^2 \end{vmatrix}$

$$= \begin{vmatrix} \frac{1}{a}(abc) & a & a^2 \\ \frac{1}{b}(abc) & b & b^2 \\ \frac{1}{c}(abc) & c & c^2 \end{vmatrix}$$

$$= (abc)\begin{vmatrix} \frac{1}{a} & \frac{a^2}{a} & \frac{a^3}{a} \\ \frac{1}{b} & \frac{b^2}{b} & \frac{b^3}{b} \\ \frac{1}{c} & \frac{c^2}{c} & \frac{c^3}{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & a^2 & a^3 \\ 1 & b^2 & b^3 \\ 1 & c^2 & c^3 \end{vmatrix}$$

17. Let $\Delta = \begin{vmatrix} a+1 & 2 & 3 \\ 1 & a+1 & 3 \\ 3 & -6 & a+1 \end{vmatrix}$

Putting a = -1 in the above, we get

$$\Delta = \begin{vmatrix} 0 & 2 & 3 \\ 1 & 0 & 3 \\ 3 & -6 & 0 \end{vmatrix}$$

$$= 0(0+18) - 2(0-9) + 3(-6-0)$$

$$= 0 + 18 - 18 = 0$$

18. L.H.S. $= \begin{vmatrix} a-b-c & 2a & 2a \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & a+b+c & a+b+c \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix} \quad (R_1 \to R_1 + R_2 + R_3)$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 2b & b-c-a & 2b \\ 2c & 2c & c-a-b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 0 & 0 \\ 2b & c-a-b & 0 \\ 2c & 0 & -a-b-c \end{vmatrix}$$

$(c_2 \to c_2 - c_1 \;\; c_3 \to c_3 - c_1)$

$$= (a+b+c)^3$$

19. Let $\Delta = \begin{vmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{vmatrix}$

Multiplying a, b, c in c_1, c_2 and c_3 respectively and dividing by abc, we get

$$\Delta = \frac{1}{abc}\begin{vmatrix} a(a^2+1) & ab^2 & ac^2 \\ a^2b & b(b^2+1) & bc^2 \\ a^2c & b^2c & c(c^2+1) \end{vmatrix}$$

Taking out a, b, c from R_1, R_2, R_3, we get

$$\Delta = \frac{abc}{abc}\begin{vmatrix} a^2+1 & b^2 & c^2 \\ a^2 & b^2+1 & c^2 \\ a^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= \begin{vmatrix} 1+a^2+b^2+c^2 & b^2 & c^2 \\ 1+a^2+b^2+c^2 & b^2+1 & c^2 \\ 1+a^2+b^2+c^2 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & b^2+1 & c^2 \\ 1 & b^2 & c^2+1 \end{vmatrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & b^2 & c^2 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} \quad \begin{matrix} (R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1) \end{matrix}$$

$$= (1+a^2+b^2+c^2)\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= 1+a^2+b^2+c^2$$

20. $\Delta = \begin{vmatrix} x+a & b & c \\ b & x+c & a \\ c & a & x+b \end{vmatrix}$

$= \begin{vmatrix} x+a+b+c & b & c \\ x+a+b+c & x+c & a \\ x+a+b+c & a & x+b \end{vmatrix} \quad (c_1 \to c_1 + c_2 + c_3)$

$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 1 & x+c & a \\ 1 & a & x+b \end{vmatrix}$

$= (x+a+b+c)\begin{vmatrix} 1 & b & c \\ 0 & x+c-b & a-c \\ 0 & a-b & x+b-c \end{vmatrix}$

$= (x+a+b+c)(x^2 - a^2 - b^2 - c^2 + ab + bc + ca)$

21. Let $\Delta = \begin{vmatrix} 3 & 6 & 9 \\ -2 & 4 & -6 \\ 8 & 16 & 24 \end{vmatrix}$

$= \begin{vmatrix} 3 & 6 & 3.3 \\ -2 & 4 & 3(-2) \\ 8 & 4 & 3.8 \end{vmatrix} = 3\begin{vmatrix} 3 & 6 & 3 \\ -2 & 4 & -2 \\ 8 & 4 & 8 \end{vmatrix} = 3x0 = 0$

22. $\begin{vmatrix} 1 & a & a^2 \\ a^2 & 1 & a \\ a & a^2 & 1 \end{vmatrix}$

$= 1(1-a^3) - a(a^2 - a^2) + a^2(a^4 - a)$

$= 1 - a^3 + a^6 - a^3$

$= 1 - 2a^3 + a^6 = (1-a^3)^2$

which is a perfect square.

23. Let $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix}$

$= \begin{vmatrix} 1 & 0 & 0 \\ a & b-a & c-a \\ a^2 & b^2-a^2 & c^2-a^2 \end{vmatrix} \quad (c_2 \to c_2 - c_1,\ c_3 \to c_3 - c_1)$

$= \begin{vmatrix} b-a & c-a \\ (b-a)(b+a) & (c-a)(c+a) \end{vmatrix}$

$= (b-a)(c-a)\begin{vmatrix} 1 & 1 \\ b+a & c+a \end{vmatrix}$

$= -(a-b)(c-a)\ (c+a-b-a)$

$= -(a-b)(c-a)\ x - (b-c)$

$= (a-b)(b-c)(c-a)$

24. $\begin{vmatrix} 15-2x & 11 & 10 \\ 11-3x & 17 & 16 \\ 7-x & 14 & 13 \end{vmatrix} = 0$

$\Rightarrow \begin{vmatrix} 15-2x & 1 & 10 \\ 11-3x & 1 & 16 \\ 7-x & 1 & 16 \end{vmatrix} = 0 \quad (c_2 \to c_2 - c_3)$

$\Rightarrow \begin{vmatrix} 15-2x & 1 & 10 \\ -4-x & 0 & 6 \\ -8+x & 0 & 3 \end{vmatrix} = 0 \quad (R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$

$\Rightarrow -1[3(-4-x) - 6(-8+x)] = 0$

$\Rightarrow -12 + 3x + 48 - 6x = 0$

$\Rightarrow -3x + 60 = 0$

$\Rightarrow x = 20$

25. $\begin{vmatrix} 17 & 58 & 97 \\ 19 & 60 & 99 \\ 18 & 59 & 98 \end{vmatrix}$

$= \begin{vmatrix} 17 & 58 & 97 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{vmatrix} \quad (R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1)$

$= 2\begin{vmatrix} 17 & 58 & 97 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 2x0 = 0$

26. $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

$\begin{vmatrix} a\left(\frac{1+a}{a}\right) & b.\frac{1}{b} & c.\frac{1}{c} \\ a.\frac{1}{a} & b\left(\frac{1+b}{b}\right) & c.\frac{1}{c} \\ a.\frac{1}{a} & b.\frac{1}{b} & c\left(\frac{1+c}{c}\right) \end{vmatrix}$

$= abc\begin{vmatrix} \frac{1}{a}+1 & \frac{1}{b} & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b}+1 & \frac{1}{c} \\ \frac{1}{a} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$

$= abc\begin{vmatrix} 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b}+1 & \frac{1}{c} \\ 1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c} & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$

$= abc\left(1+\frac{1}{b}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 1 & \frac{1}{b}+1 & \frac{1}{c} \\ 1 & \frac{1}{b} & \frac{1}{c}+1 \end{vmatrix}$

$= abc\left(1+\frac{1}{b}+\frac{1}{b}+\frac{1}{c}\right)\begin{vmatrix} 1 & \frac{1}{b} & \frac{1}{c} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$ $(R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1)$

$= abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$

27. $\text{LHS} = \begin{vmatrix} 1 & a & a^2-bc \\ 1 & b & b^2-ca \\ 1 & c & c^2-ab \end{vmatrix}$

$= \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & bc \\ 1 & b & ca \\ 1 & c & ab \end{vmatrix}$

$= (a-b)(b-c)(c-a) - (a-b)(b-c)(c-a)$

$= 0$

28. $\text{LHS} = \begin{vmatrix} 1 & bc & a(b+c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix}$

$= \begin{vmatrix} 1 & bc & ab+ac \\ 1 & ca & bc+ab \\ 1 & ab & ac+bc \end{vmatrix}$

$= \begin{vmatrix} 1 & bc & ab+bc+ca \\ 1 & ca & ab+bc+ca \\ 1 & ab & ab+bc+ca \end{vmatrix}$ $(C_3 \to C_3 + C_2)$

$= (ab+bc+ca)\begin{vmatrix} 1 & bc & 1 \\ 1 & ca & 1 \\ 1 & ab & 1 \end{vmatrix}$

$= (ab+bc+ca) \times 0 = 0$

29. $\text{LHS} = \begin{vmatrix} x & y & z \\ x^2 & y^2 & z^2 \\ x^3 & y^3 & z^3 \end{vmatrix}$

$= xyz\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$

$= xyz(x-y)(y-z)(z-x)$ [As No. 23]

30. $\text{LHS} = \begin{vmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{vmatrix}$

$= \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ bc & ca & ab \end{vmatrix}$

$$= \frac{1}{abc}\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ abc & abc & abc \end{vmatrix}$$

$$= \frac{1}{abc} . abc\begin{vmatrix} a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \\ 1 & 1 & 1 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ a^2 & b^2 - a^2 & c^2 - a^2 \\ a^3 & b^3 - a^3 & c^3 - a^3 \end{vmatrix}$$

$$= \begin{vmatrix} b^2 - a^2 & c^2 - a^2 \\ b^3 - a^3 & c^3 - a^3 \end{vmatrix}$$

$$= \begin{vmatrix} (b-a)(b+a) & (c-a)(c+a) \\ (b-a)(b^2+ab+a^2) & (c-a)(c^2+ca+a^2) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} b+a & c+a \\ b^2+ab+a^2 & c^2+ca+a^2 \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} b+a & (c+a)-(b+a) \\ b^2+ab+a^2 & (c^2+ca+a^2)-(b^2+ab+a^2) \end{vmatrix}$$

$$= (b-a)(c-a)\begin{vmatrix} b+a & c-b \\ b^2+ab+a^2 & (c-b)-(c+b+a) \end{vmatrix}$$

$$= (b-a)(c-a)(c-b)\begin{vmatrix} b+a & 1 \\ b^2+ab+a^2 & a+b+c \end{vmatrix}$$

$$= (a-b)(b-c)(c-a)\ (ab+bc+ca).$$

31. Given that P(A) = 0.6, P(B)=0.5.

$P(A \cap B) = 0.2$

$$P(A/B^C) = \frac{P(A \cap B^C)}{P(B^C)} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$$

$$= \frac{0.6 - 0.2}{1 - 0.5} = \frac{0.4}{0.5} = \frac{4}{3}$$

32. A pair of dice i thrown.

Let S be the sample space.

$|S| = 36$

Let A be the event of getting atleast 9 and B be the event where 5 appears on atleast one of the dice.

A = {(3, 6), (4, 5), (5, 4), (6, 3), (4, 6), (5, 5), (6, 4), (5, 6), (6, 5), (6, 6) }

B = {(1, 5), (2, 5), (3, 5), (4, 5), (5, 5), (6, 5), (5, 1), (5, 2), (5, 3), (5, 4), (5,6)}

$A \cap B = \{(4,5),(5,4),(5,5),(5,6),(6,5)\}$

P (of getting of atleast 9 if 5 appears on at least of the dice)

$$= P(A/B) = \frac{P(A \cap B)}{P(B)} = \frac{\frac{5}{36}}{\frac{11}{36}} = \frac{5}{11}.$$

33. Two dice are thrown.

Let S be the sample space.

$|S| = 36$.

Let A be the event that one of the dice is 3.

A= {(3,1), (3,2), (3,3), (3,4), (3,6), (1,3), (2,3), (4,3), (5,3), (6,3)}

Let B be the event that the sum is 5

B = {(1, 4), (4, 1), (2, 3), (3, 2)}

$A \cap B = \{(2,3),(3,2)\}$

$\therefore |A \cap B| = 2, \quad |B| = 4$

$$P|A/B| = \frac{P(A \cap B)}{P(B)} = \frac{\frac{2}{36}}{\frac{4}{36}} = \frac{2}{4} = \frac{1}{2}.$$

34. Given that P(A) = 0.3, P(B) = 0.4 and

$P(A \cup B) = 0.6$

We know $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow 0.6 = 0.3 + 0.4 - P(A \cap B)$

$\Rightarrow P(A \cap B) = 0.1$

(i) $P(A/B^C) = \frac{P(A \cap B^C)}{P(B^C)} = \frac{P(A) - P(A \cap B)}{1 - P(B)}$

$= \frac{0.3 - 0.1}{1 - 0.4} = \frac{0.2}{0.6} = \frac{2}{6} = \frac{1}{3}$

(ii) $P(B/A) = \frac{P(B \cap A)}{P(A)} = \frac{0.1}{0.3} = \frac{1}{3}$

35. Given that P(A)=0.3 and P(B)=0.6.

A and B are independent events.

P(A and B) $= P(A \cap B) = P(A).P(B)$

$= 0.3 \times 0.6 = 0.18$

(i) P(A and not B) = $P(A \cap B^C) = P(A).P(B^C)$

[$\because$ A and B are independent, so A and B^C are independent.]

$= P(A)[1 - P(B)]$

$= 0.3[1 - 0.6] = 0.12$

(ii) P (A or B) $= P(A \cup B)$

$= P(A) + P(B) - P(A \cap B)$

$= 0.3 + 0.6 - 0.18 = 0.72$

(iii) P (neither A nor B) $= P(A^C \text{ or } B^C)$

$= P(A^C \cup B^C) = P(A \cap B)^C$

$= 1 - P(A \cup B)$

$= 1 - 0.72 = 0.28$

36. The given distribution is a probability distribution. So the sum of the probaility is 1.

$0 + 2p + 2p + 3p + p^2 + 2p^2 + 7p^2 + 2p = 1$

$\Rightarrow 10p^2 + 9p - 1 = 0$

$\Rightarrow (10p - 1)(p + 1) = 0$

$\Rightarrow p = \frac{1}{10}$

($\because$ p=-1 is rejected)

37. There are 9 digits, 1, 2, 3, 4, 5, 6, 7, 8, 9.

Two different digits are selected.

Let S be the sample space.

$|S| = 9 \times 8 = 72$

We shall find the probability that 3 is one of the numbers selected if the sum is even.

Let A be the event where 3 is one of the numbers selected and B be the event where the sum of the numbers is even.

A = {(1,3), (2,3), (4,3), (5,3), (6,3), (7,3), (8,3), (9,3) (3,1), (3,2), (3,4), (3,5), (3,6), (3,7), (3,8),(3,9)}

$\therefore |A| = 16$

A = {(1,3), (1,5), (1,7), (1,9), (2,4), (2,6), (2,8), (3,1), (3,5), (3,7), (3,9), (4,2), (4,6), (4,8), (5,1), (5,3), (5,7), (5,9), (6,2), (6,4), (6,8), (7,1), (7,3), (7,5), (7,9), (8,2), (8,4),(8,6), (9,1), (9,3), (9,5), (9,7)}

$|B| = 32$

$A \cap B$ = {(1,3), (5,3), (7,3), (3,1), (3,5), (3,7), (3,9)}

$|A \cap B| = 8$

Required Probability = P (A/B)

$= \frac{P(A \cap B)}{P(B)} = \frac{\frac{8}{72}}{\frac{32}{72}} = \frac{8}{32} = \frac{1}{4}.$

38. Given that n = 6

Also given that 4p (x = 4) = p (x = 2)

$\Rightarrow 4.{}^nc_4\, p^4 q^{n-4} = {}^n c_2\, p^2 q^{n-2}$ where $p + q = 1$.

$\Rightarrow 4.{}^6c_4\, p^4 q^2 = {}^6c_2 p^2 q^4$

$\Rightarrow 4.\frac{6.5.4.3}{4.3.2.1}.p^4q^2 = \frac{6.5}{2.1}p^2q^4$

$\Rightarrow 4p^4q^2 - p^2q^4 = 0$

$\Rightarrow p^2q^2(4p^2 - q^2) = 0$

$\Rightarrow p^2(1 - p^2)[4p^2 - (1 - p)^2] = 0$

$\Rightarrow p^2(1 - p^2)(4p^2 - 1 + 2p^2) = 0$

$\Rightarrow p^2(1 - p^2)(3p^2 + 2p - 1) = 0$

$\Rightarrow p^2(1 - p^2)(p + 1)(3p - 1) = 0$

$\Rightarrow p = 0, 1, \frac{1}{3}$ (p = -1 is rejected)

39. Let X be the number of heads when a coin is tossed three times.

S={HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

X can take the values 0, 1, 2 and 3.

P (X = 0) = P (no head occurs) $=\frac{1}{8}$

P (X = 1) = P (one head occurs) $=\frac{3}{8}$

P (X = 2) = P (two heads occur) $=\frac{3}{8}$

P (X = 3) = P (three heads occur) $=\frac{1}{8}$

Thus the probability distribution is as follows

X	0	1	2	3
P(X)	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

For finding mean,

X	P(X)	xi pi
0	$\frac{1}{8}$	0
1	$\frac{3}{8}$	$\frac{3}{8}$
2	$\frac{3}{8}$	$\frac{6}{8}$
3	$\frac{1}{8}$	$\frac{3}{8}$

Mean = $\sum x_i p_i = 0+\frac{3}{8}+\frac{6}{8}+\frac{3}{8}=\frac{12}{8}=\frac{3}{2}$

40. Given that $P(A)=0.6, P(B/A)=0.5$

Given that A and B are independent events.

$P(A\cap B)=P(A)\ .\ P(B).$ ... (1)

We know $P(B/A)=\frac{P(B\cap A)}{P(A)}$

$\Rightarrow 0.5=\frac{P(A\cap B)}{0.6}$

$\Rightarrow P(A\cap B)=0.5\ x\,0.6=0.3$

From (1) we get

0.3 = 0.6 x P(B).

$\Rightarrow P(B)=\frac{0.3}{0.6}=\frac{3}{6}=\frac{1}{2}$

$P(A\cup B)=P(A)+P(B)-P(A\cap B)$

=0.6 + 0.5 - 0.3 = 0.8

41. Let $\Delta=\begin{vmatrix} b^2-ab & b-c & bc-ac \\ ab-a^2 & a-b & b^2-ab \\ bc-ac & c-a & ab-a^2 \end{vmatrix}$

$=\begin{vmatrix} b(b-a) & b-c & c(b-a) \\ a(b-a) & a-b & b(b-a) \\ c(b-a) & c-a & a(b-a) \end{vmatrix}$

$=(b-a)^2\begin{vmatrix} b & b-c & c \\ a & a-b & b \\ c & c-a & a \end{vmatrix}$

$=(b-a)^2\begin{vmatrix} b & b-c+c & c \\ a & a-b+b & b \\ c & c-a+a & a \end{vmatrix}$ $(C_2\to C_2+C_3)$

$=(b-a)^2\begin{vmatrix} b & b & c \\ a & a & b \\ c & c & a \end{vmatrix}$

$=(b-a)^2 x 0=0$

(As 1st & 2nd columns are identical)

42. Let $\Delta=\begin{vmatrix} a^2 & bc & c^2+ac \\ a^2+ab & b^2 & ca \\ ab & b^2+bc & c^2 \end{vmatrix}$

$=abc\begin{vmatrix} a & c & c+a \\ a+b & b & a \\ b & b+c & c \end{vmatrix}$

$=abc\begin{vmatrix} 0 & c & a+c \\ 2b & b & a \\ 2b & b+c & c \end{vmatrix}$ $[C_1\to C_1+C_2-C_3]$

$=2ab^2c\begin{vmatrix} 0 & c & a+c \\ 1 & b & a \\ 1 & b+c & c \end{vmatrix}$

$$= 2ab^2c\begin{vmatrix} 0 & c & a+c \\ 0 & -c & a-c \\ 1 & b+c & c \end{vmatrix} (R_2 \to R_2 - R_3)$$

$$= 2ab^2c\,[c(a-c)+c(a+c)]$$

$$= 4a^2b^2c^2 \cdot$$

43. Given that

x = cy + bz

y = az + cx

z = bx + ay

$$\Rightarrow \quad x - cy - bz = 0$$

$$cx - y + az = 0$$

$$bx + ay - z = 0$$

Eliminating x, y and z from the above, we get

$$\begin{vmatrix} 1 & -c & -b \\ c & -1 & a \\ b & a & -1 \end{vmatrix} = 0$$

$$\Rightarrow 1(1-a^2)-(-c)(-c-ab)+(-b)(ac+b)=0$$

$$\Rightarrow a^2+b^2+c^2+2abc=1$$

44. Let $\Delta = \begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+y & -2c \end{vmatrix}$ ...(1)

Putting a+b=0, i.e. b = -a (1), we get

$$\Delta = \begin{vmatrix} -2a & 0 & c+a \\ 0 & 2a & c-a \\ c+a & c+a & -2c \end{vmatrix}$$

$$= -2a\left[2a(-2c)-(c-a)^2\right] - 0 + (c+a)$$

$$[0-2a(c+a)]$$

$$= 2a[(c-a)^2+4ac]-2a(c+a)^2$$

$$= 2a(c+a)^2 - 2a(c+a)^2 = 0$$

$\therefore$ a + b is a factor of Δ.

Similalry b + c and c + a are factors of Δ.

Let $\Delta = k(a+b)(b+c)(c+a)$

$$\Rightarrow \begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix}$$

$$= k(a+b)(b+c)(c+a) \quad \text{... (2)}$$

Putting a = 0, b = 1, c = 1, in the above, we get

$$\begin{vmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -2 \end{vmatrix} = k.\,1\,.\,2\,.\,1$$

$$\Rightarrow 2k = 0 - 1(-2-2) + 1(2+2)$$

$$\Rightarrow 2k = 8$$

$$\Rightarrow k = 4 \cdot$$

From (2), we get

$$\begin{vmatrix} -2a & a+b & c+a \\ a+b & -2b & b+c \\ c+a & c+b & -2c \end{vmatrix} = 4(a+b)(b+c)(c+a).$$

45. The given equation is

$$\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ x-4 & 2x-9 & 3x-16 \\ x-8 & 2x-27 & 3x-64 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ -2 & -6 & -12 \\ -6 & -24 & -60 \end{vmatrix} = 0 \quad \begin{matrix} (R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1) \end{matrix}$$

$$\Rightarrow (-2)(-6)\begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & 2x-3 & 3x-4 \\ 1 & 3 & 6 \\ 1 & 4 & 10 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x-2 & 1 & 2 \\ 1 & 1 & 3 \\ 1 & 2 & 7 \end{vmatrix} = 0 \quad \begin{matrix} [C_2 \to C_2 - 2C_1 \\ C_3 \to C_3 - 3C_1] \end{matrix}$$

$$\Rightarrow (x-2)(7-6) - 1(7-3) + 2(2-1) = 0$$

$$\Rightarrow x - 2 - 4 + 2 = 0$$

$$\Rightarrow x = 4 \cdot$$

4. Continuity and Differentiability, Application of Derivatives

1. Given fucntion is

$$f(x)=\begin{cases} ax^2+b & \text{if} \quad x<1 \\ 1 & \text{if} \quad x=1 \\ 2ax-b & \text{if} \quad x>1 \end{cases}$$

L.HL = $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^-} ax^2+b=a+b$

R.HL = $\lim\limits_{x\to 1^+} f(x)=\lim\limits_{x\to 1^+} 2ax-b=2a-b$

Again f(1) = 1

Since the function is continuous at x = 1,

we have $\lim\limits_{x\to 1^-} f(x)=\lim\limits_{x\to 1^+} f(x)=f(1)$

$\Rightarrow$ a + b = 2a – b = 1

$\Rightarrow$ a + b = 1 ....(1)

e 2a – b = 1 ....(2)

Solving (1) and (2), we get $a=\frac{1}{3}, b=\frac{2}{3}$

2. Let f(x) = sin x.

Let $x_1 \in R$ be any point and $\in$ be any arbitary positive number.

$|f(x)-f(x_1)|=|\sin x-\sin x_1|$

$= \left|2\cos\frac{x+x_1}{2}.\sin\frac{x-x_1}{2}\right|$

$= 2\left|\cos\frac{x+x_1}{2}\right|\left|\sin\frac{x-x_1}{2}\right|$ ....(1)

We know for every value of x_1, $\left|\cos\frac{x+x_1}{2}\right|\le 1$

From elementary trigonometry, we have

$\left|\sin\frac{x-x_1}{2}\right|\le\left|\frac{x-x_1}{2}\right|$

From (1), we get

$|f(x)-f(x_1)|\le 2.1.\left|\frac{x-x_1}{2}\right|$

$\Rightarrow |f(x)-f(x_1)|\le|x-x_1|$

When $|x-x_1| < \in$ then $|f(x)-f(x_1)|<\in$

$\Rightarrow \lim\limits_{x\to x_1}$ sin x = sin x_1

$\Rightarrow$ The function is continuous at x = x_1.

3. Given function is

$$f(x)=\begin{cases} 2x-1 & \text{when} \quad x<2 \\ k & \text{when} \quad x=2 \\ x+1 & \text{when} \quad x>2 \end{cases}$$

L.HL = $\lim\limits_{x\to 2^-} f(x)=\lim\limits_{h\to \nu} f(2-h)$

$= \lim\limits_{h\to 0} 2(2-h)-1=\lim\limits_{h\to 0} 4-2h-1$

$= \lim\limits_{h\to 0} 3-2h=3$.

R.HL = $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{h\to 0} +(2+h)=\lim\limits_{h\to 0} (2+h)+1$

$= \lim\limits_{h\to 0} (3+h)=3$

Also f(2) = K.

Since the function is continuous at x=2,

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 2^+} f(x)=f(2)$

$\Rightarrow$ 3x = 3 = K $\Rightarrow$ K = 3

4. Given function is

$$f(x)=\begin{cases} kx+1 & \text{if } x\le\pi \\ \cos x & \text{if } x>\pi \end{cases}$$

L.H.L = $\lim\limits_{x\to \pi^-} f(x)$

$= \lim\limits_{h\to 0} f(\pi-h) = \lim\limits_{h\to 0} K(\pi-h)+1$

$= K\pi+1$

R.H.L = $\lim\limits_{x\to \pi^+} f(x)=\lim\limits_{h\to 0} f(\pi+h)$

$= \lim\limits_{h\to 0} \cos(\pi+h)=\lim\limits_{h\to 0} -\cos h=-1$.

$f(\pi)=K\pi+1$

Since the function is continuous at $x=\pi$,

we have $\lim\limits_{x\to \pi^-} f(x)=\lim\limits_{x\to \pi^+} f(x)=f(\pi)$

$\Rightarrow K\pi+1=-1$

$\Rightarrow K\pi+1=-2$

$\Rightarrow K=-\frac{2}{\pi}$

5. The given function is

$$f(x)=\begin{cases}\dfrac{1-\cos 4x}{8x^2} & \text{if } x\neq 0\\ k & \text{if } x=0\end{cases}$$

Given that the function is continuous at x=0.

$\therefore \quad \lim_{x\to 0} f(x)=f(0)$ .....(1)

$\Rightarrow \lim_{x\to 0}\dfrac{1-\cos 4x}{8x^2}=K$

$\Rightarrow \lim_{x\to 0}\dfrac{2\sin^2 2x}{8x^2}=K$

$\Rightarrow \lim_{x\to 0}\left(\dfrac{\sin 2x}{2x}\right)^2=K$

$\Rightarrow$ 1 = K $\Rightarrow$ K = 1

6. Give that $x=a\cos^3\theta,\quad y=a\sin^3\theta$.

$\dfrac{dx}{d\theta}=a\dfrac{d\cos^3\theta}{d\theta}=a\dfrac{d\cos^3\theta}{d\cos\theta}.\dfrac{d\cos\theta}{d\theta}$

$= a.\,3\cos^2\theta(-\sin\theta)$

$=-3a\cos^2\theta.\sin\theta$

$\dfrac{dy}{d\theta}=a\dfrac{d\sin^3\theta}{d\theta}=a\dfrac{d\sin^3\theta}{d\sin\theta}.\dfrac{d\sin\theta}{d\theta}$

$= 3a\sin^2\theta\cos\theta$.

$$\frac{dy}{dx}=\frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}=\frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta.\sin\theta}=-\tan\theta$$

$$\frac{d^2y}{dx^2}=-\frac{d\tan\theta}{dx}=-\frac{d\tan\theta}{d\theta}.\frac{d\theta}{dx}$$

$$=-\sec^2\theta\times\frac{1}{-3a\cos^2\theta.\sin\theta}=\frac{1}{3a}\frac{1}{\cos^4\theta.\sin\theta}$$

When $\theta=\dfrac{\pi}{6}$, $\dfrac{d^2y}{dx^2}=\dfrac{1}{3a\cos^4\frac{\pi}{6}.\sin\frac{\pi}{6}}$

$$=\frac{1}{3a.\left(\frac{\sqrt{3}}{2}\right)^4.\frac{1}{2}}$$

$$=\frac{32}{27a}.$$

7. Given that

x sin (a + y) + sin a cos (a + y) = 0

$\Rightarrow$ x sin (a + y) = – sina cos(a+y)

$\Rightarrow$ x = – sin a. cot (a+y).

Differentiative both sides w.r.t y, we get

$$\frac{dx}{dy}=-\sin a\frac{d\cot(a+y)}{dy}$$

$$=\sin a.x-\operatorname{cosec}^2(a+y)$$

$$=\frac{\sin a}{\sin^2(a+y)}$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

8. Given that $x^m.y^n=\left(\dfrac{x}{y}\right)^{m+n}$

$\Rightarrow \log(x^m.\,y^n)=\log\left(\dfrac{x}{y}\right)^{m+n}$

$\Rightarrow$ m log x + n log y = (m+n)(log x-log y)

$\Rightarrow$ (m+2n) log y = n log x.

Differentiative both sides w.r.t. x, we get

$(m+2n)\,\dfrac{1}{y}.\dfrac{dy}{dx}=\dfrac{n}{x}$

$\Rightarrow \dfrac{dy}{dx}=\dfrac{ny}{(m+2n)x}$

9. Given that $y=\sin^{-1}\left(\dfrac{2\sqrt{t}-1}{t^2}\right)$

$$\frac{dy}{dt}=\frac{d\sin^{-1}\left(\frac{2\sqrt{t}-1}{t^2}\right)}{dt}$$

$$=\frac{d\sin^{-1}\left(\frac{2\sqrt{t}-1}{t^2}\right)}{d\left(\frac{2\sqrt{t}-1}{t^2}\right)}.d\left(\frac{\frac{2\sqrt{t}-1}{t^2}}{dt}\right)$$

$$= \frac{1}{\sqrt{1-\left(\frac{2\sqrt{t}-1}{t^2}\right)^2}} \times \frac{t^2.d\left(2\sqrt{t}-1\right)-\left(2\sqrt{t}-1\right)\frac{dt^2}{dt}}{t^4}$$

$$= \frac{t^2}{\sqrt{t^4-4t+4\sqrt{t}-1}} \times \frac{t\sqrt{t}-4\ \ t\ \sqrt{t}+2t}{t^4}$$

$$= \frac{2t-3t\sqrt{t}}{t^2\sqrt{t^4-4t+4\sqrt{t}-1}}.$$

10. Given that $x = a\sec\theta$, $y = b\tan\theta$.

$$\frac{dx}{d\theta} = a\sec\theta.\tan\theta$$

$$\frac{dy}{d\theta} = b\sec^2\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dx}}{\frac{dx}{d\theta}} = \frac{b}{a}\operatorname{cosec}\theta$$

$$\frac{d^2y}{dx^2} = \frac{b}{a}\frac{d\operatorname{cosec}\theta}{d\theta}.\frac{d\theta}{dx}$$

$$= \frac{b}{a}\left(-\operatorname{cosec}\theta.\cot\theta\right).\frac{1}{a\sec\theta\tan\theta}$$

$$= -\frac{b^4}{a^2y^3}.$$

11. $x^y = e^{x-y}$

Taking logarthim of both sides, we get

$y\log x = x - y$

$\Rightarrow y(1+\log x) = x$

$$\Rightarrow y = \frac{x}{1+\log x}$$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{x}{1+\log x}\right)$$

$$= \frac{(1+\log x)\frac{dx}{dx} - x\frac{d(1+\log x)}{dx}}{(1+\log x)^2}$$

$$= \frac{\log x}{(1+\log x)^2} = \frac{\log x}{(\log ex)^2}$$

12. Sin y = x sin (a+y) $\Rightarrow x = \frac{\sin y}{\sin(a+y)}$

Differentiating both sides w.r.t. x, we get

$$\frac{d\sin y}{dx} = \frac{d\left(x\sin(a+y)\right)}{dx}$$

$$\Rightarrow \cos y\frac{dy}{dx} = \sin(a+y) + x\cos(a+y).\frac{dy}{dx}$$

$$\Rightarrow \left[\cos y - x\cos(a+y)\right]\frac{dy}{dx} = \sin(a+y)$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sin(a+y)}{\cos y - x\cos(a+y)}$$

$$= \frac{\sin(a+y)}{\cos y - \frac{\sin y}{\sin(a+y)}\cos(a+y)}$$

$$= \frac{\sin^2(a+y)}{\sin a}.$$

13. Given that x = a (cos + t sin t)

y = a (sin t – t cos t)

$$\frac{dx}{dt} = a\frac{d(\cos t + t\sin t)}{dt} = a t\cos t$$

$$\frac{dy}{dt} = a\frac{d(\sin t - t\cos t)}{dt} = a t\sin t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \tan t$$

$$\frac{d^2y}{dx^2} = \frac{d\tan t}{dx} = \frac{d\tan t}{dt}.\frac{dt}{dx}$$

$$= \sec^2 t \times \frac{1}{a t\cos t} = \frac{\sec^3 t}{dt}$$

14. Given that $\cos x = \sqrt{\frac{1}{1+t^2}}$, $\sin y = \frac{2t}{1+t^2}$

$\Rightarrow \cos x = \sqrt{\frac{1}{1+t^2}}, \; y = \sin^{-1}\frac{2t}{1+t^2}$.

Let t = $\tan\theta \Rightarrow \theta = \tan^{-1} t$

$\therefore x = \cos^{-1}\frac{1}{\sqrt{1+\tan^2\theta}} = \cos^{-1}\cos\theta = \theta = \tan^{-1} t$

$$\frac{dx}{dt} = \frac{1}{1+t^2}$$

$$y = \sin^{-1}\frac{2t}{1+t^2} = 2\tan^{-1} t$$

$$\frac{dy}{dt} = \frac{2}{1+t^2}$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{\frac{1}{1+t^2}}{\frac{2}{1+t^2}} = \frac{1}{2}$$

15. $y^x = x^{\sin y}$

$\Rightarrow \log y^x = \log x^{\sin y}$

$\Rightarrow x \log y = \sin y . \log x$

Differentiating both sides w.r.t. x, we get

$$\frac{d(x\log y)}{dx} = \frac{d\sin y \log x}{dx}$$

$$\Rightarrow x.\frac{d\log y}{dx} + \log y.\frac{dx}{dx}$$

$$= \sin y\frac{d\log x}{dx} + \log x\frac{d\sin y}{dx}$$

$$\Rightarrow x.\frac{1}{y}.\frac{dy}{dx} + \log y$$

$$= \sin y.\frac{1}{x} + \log x.\log y\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(\sin y - x\log y)}{x(x - y\log x\cos y)}$$

16. $y^2 \cot x = x^2 \cot y$

Differentiating both sides with respect to x,

we get $\frac{d(y^2\cot x)}{dx} = \frac{d(x^2\cot y)}{dx}$

$$\Rightarrow y^2\frac{d\cot x}{dx} + \cot x.\frac{dy^2}{dx}$$

$$= x^2\frac{d\cot y}{dx} + \cot y\frac{dx^2}{dx}$$

$$\Rightarrow y^2(-\operatorname{cosec}^2 x) + \cot x.2y\frac{dy}{dx}$$

$$= x^2(-\operatorname{cosec}^2 y)\frac{dy}{dx} + 2x\cot y$$

$$\Rightarrow \frac{dy}{dx} = \frac{2x\cot y + y^2\operatorname{cosec}^2 x}{2y\cot x + x^2\operatorname{cosec}^2 y}$$

17. Let $y = x^{\sin x}$

$\therefore \log y = \log_x^{\sin x}$

$$\frac{d\log y}{dx} = \frac{d\sin x\log x}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx} = \sin x.\frac{1}{x} + \log x.\cos x$$

$$\Rightarrow \frac{dy}{dx} = y\left(\frac{\sin x}{x} + \cos x.\log x\right)$$

$$= x^{\sin x}\left(\frac{\sin x}{x} + \cos x\log x\right)$$

18. Let $y = \sin^{-1}\frac{2x}{1+x^2}$, $z = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

We shall find out $\frac{dy}{dz}$.

Let x = $\tan\theta \Rightarrow \theta = \tan^{-1} x$.

$$y = \sin^{-1}\frac{2x}{1+x^2} = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}\sin 2\theta$$

$$= 2\theta = 2\tan^{-1} x$$

$$\therefore \frac{dy}{dx} = \frac{2d\tan^{-1}x}{dx} = \frac{2}{1+x^2}$$

$$z=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=2\tan^{-1}x$$

$$\frac{dz}{dx}=2\frac{d\tan^{-1}x}{dx}=\frac{2}{1+x^2}$$

$$\frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=\frac{\frac{2}{1+x^2}}{\frac{2}{1+x^2}}=1$$

19. Let $y=\tan^{-1}\left(\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}\right)$

let $x^2=\cos\theta\Rightarrow\theta=\cos^{-1}x^2$.

$$y=\tan^{-1}\left(\frac{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}\right)$$

$$=\tan^{-1}\left(\frac{\sqrt{2}\cos\frac{\theta}{2}+\sqrt{2}\sin\theta}{\sqrt{2}\cos\frac{\theta}{2}-\sqrt{2}\sin\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\left(\frac{1+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}\right)=\tan^{-1}\left(\frac{\tan\frac{\pi}{4}+\tan\frac{\theta}{2}}{1=\tan\frac{\pi}{4}\tan\frac{\theta}{2}}\right)$$

$$=\tan^{-1}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\frac{\pi}{4}+\frac{1}{2}\theta$$

$$=\frac{\pi}{4}+\frac{1}{2}\cos^{-1}x^2$$

$$\frac{dy}{dx}=\frac{d\left(\frac{\pi}{4}\right)}{dx}+\frac{1}{2}\frac{d\cos^{-1}x^2}{dx}$$

$$=U+\frac{1}{2}\times\frac{d\cos^{-1}x^2}{dx^2}.\frac{dx^2}{dx}$$

$$=\frac{1}{2}\times-\frac{1}{\sqrt{1-x^4}}\times 2x=-\frac{x}{\sqrt{1-x^4}}$$

20. Given that y = (sin y)$^{\sin 2x}$

log y = log (Isin y)$^{\sin 2x}$ = sin 2x.log (sin y)

Differentiating both sides w.r.t. x, we get

$$\frac{d\log y}{dx}=\frac{d\sin 2x\log(\sin y)}{dx}$$

$$\Rightarrow\frac{1}{y}.\frac{dy}{dx}=\sin 2y.\frac{1}{\sin y}.\cos y\frac{dy}{dx}+2\cos 2x\log(\sin y)$$

$$=\sin 2y.\cot y\frac{dy}{dx}+2\cos 2x\log(\sin y)$$

$$\Rightarrow\left(\frac{1}{y}-\sin 2y\cot y\right)\frac{dy}{dx}=2\cos 2x\log(\sin y)$$

$$\Rightarrow\frac{dy}{dx}=\frac{2\cos 2x\log(\sin y)}{\frac{1}{y}-\sin 2y\cot y}$$

21. Given function is

$$f(x)=\begin{cases}\frac{1-e^{-x}}{x} & \text{when } x\neq 0\\ 1 & \text{when } x=0\end{cases}.$$

First we shall test the differentiability at x = 0.

$$\text{L.H.D}=\lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$$

$$=\lim_{h\to 0}\frac{f(0-h)-f(0)}{-h}$$

$$=\lim_{h\to 0}\frac{\frac{1-e^{n}-1}{-h}}{-h}$$

$$=\lim_{h\to 0}\frac{1-e^{h}+h}{h^2}\quad(\text{from is }\frac{0}{0})$$

$$=\lim_{h\to 0}\frac{-e^{h}+1}{2h}\quad(\text{form is }\frac{0}{0})$$

$$=\lim_{h\to 0}\frac{-e^{h}}{2}\quad\text{(Applying L' Hospital's rule)}$$

$$=-\frac{1}{2}$$

R.HD = $\lim_{x\to 0^+} \frac{f(x)-f(0)}{oth}$

$= \lim_{h\to 0} \frac{f(oth)-f(0)}{oth}$

$= \lim_{h\to 0} \frac{f(h)-f(0)}{h}$

$= \lim_{h\to 0} \frac{\frac{1-e^{-h}}{h}-1}{h} = \lim_{h\to 0} \frac{1-e^{-h}-h}{h^2}$

$= -\frac{1}{2}$

At x=0, L.H.D = R.H.D.

So the function is differentiable at x = 0.

$\Rightarrow$ The function is containing at x = 0

(Every differeniable function is continuous).

22. Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right)$, $z = \sqrt{1-x^2}$

Let $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

$y = \sec^{-1}\left(\frac{1}{2\cos^2\theta}\right) = \sec^1 \sec\theta$

$= 2\theta = 2\cos^{-1} x$

$\frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

$\frac{dz}{dx} = \frac{d\sqrt{1-x^2}}{dx} = -\frac{x}{\sqrt{1-x^2}}$

$\therefore \frac{dy}{dz} = \frac{\frac{dy}{dx}}{\frac{dz}{dx}} = \frac{-\frac{2}{\sqrt{1-x^2}}}{-\frac{x}{\sqrt{1-x^2}}} = \frac{2}{x}$

23. $y = x + \cfrac{1}{x + \cfrac{1}{x + \cfrac{1}{x + \ldots\ldots \text{to } \infty}}}$

$\Rightarrow y = x + \frac{1}{y} = \frac{xy+1}{y}$

$\Rightarrow y^2 = xy + 1.$

Differentiative both sides w.r.t x, we get

$\frac{dy^2}{dx} = \frac{d(xy+1)}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{y}{2y-x}.$

24. The equation of the curve is

$x = 2(\theta - \sin\theta),\ y = 2\,(1-\cos\theta)$

$\frac{dx}{d\theta} = 2(1-\cos\theta)$

$\frac{dy}{d\theta} = 2\sin\theta$

$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{2\sin\theta}{2(1-\cos\theta)} = \frac{\sin\theta}{1-\cos\theta}$

At $\theta = \frac{\pi}{4}, \frac{dy}{dx} = \frac{\sin\frac{\pi}{4}}{1-\cos\frac{\pi}{4}} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{2}} = \sqrt{2}+1.$

Required slope = $\sqrt{2}+1.$

25. Given equation is

$\cos y = x\cos(a+y)$

Differentiating both sides w.r.t x, we get

$\frac{d\cos y}{dx} = \frac{dx\cos(a+y)}{dx}$

$\Rightarrow \frac{d\cos y}{dy}.\frac{dy}{dx}$

$= x\frac{d\cos(a+y)}{d(a+y)}.\frac{d(a+y)}{dx} + \cos(a+y).$

$\Rightarrow -\sin y.\frac{dy}{dx}$

$= x \times -\sin(a+y).\frac{dy}{dx} + \cos(a+y)$

$\Rightarrow \left[x\sin(a+y) - \sin y\right]\frac{dy}{dx} = \cos(a+y)$

$$\Rightarrow \left[\frac{\cos y}{\cos(a+y)}.\sin(a+y)-\sin y\right]\frac{dy}{dx}$$

$$=\cos(a+y)$$

$$\Rightarrow \frac{\cos(a+y-y)}{\cos(a+y)}\frac{dy}{dx}=\cos(a+y)$$

$$\Rightarrow \frac{dy}{dx}=\frac{\cos^2(a+y)}{\cos a}$$

26. y = tan (x + y)

Differentiating both sides w.r.t x, we get

$$\frac{dy}{dx}=\frac{d\tan(x+y)}{dx}$$

$$=\frac{d\tan(x+y)}{d(x+y)}.\frac{d(x+y)}{dx}$$

$$=\sec^2(x+y)\left[1+\frac{dy}{dx}\right]$$

$$\Rightarrow \frac{dy}{dx}\left[1-\sec^2(x+y)\right]=\sec^2(x+y)$$

$$\Rightarrow \frac{dy}{dx}=\frac{\sec^2(x+y)}{1-\sec^2(x+y)}$$

$$=\frac{1+\tan^2(x+y)}{-\tan^2(x+y)}=-\frac{1+y^2}{y^2}$$

27. Let $y=\sin^{-1}\left(\frac{2x^3}{1+x^6}\right)$

Let $x^3=\tan\theta \Rightarrow \theta=\tan^{-1}x^3$

$$\therefore \quad y=\sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$=\sin^{-1}\sin 2\theta=2\theta=2\tan^{-1}x^3$$

$$\frac{dy}{dx}=2.\frac{d\tan^{-1}x^3}{dx}=2\frac{d\tan^{-1}x^3}{dx^3}.\frac{dx^3}{dx}$$

$$=\frac{2}{1+x^6}\times 3x^2=\frac{6x^2}{1+x^6}$$

28. Given function is $\tan^{-1}(\cos^2 x)$.

$$\frac{d\tan^{-1}(\cos^2 x)}{dx}=\frac{d\tan^{-1}(\cos^2 x)}{d(\cos^2 x)}.\frac{d\cos^2 x}{d\cos x}.\frac{d\cos x}{dx}$$

$$=\frac{1}{1+\cos^4 x}.2\cos x(-\sin x)$$

$$=-\frac{\sin 2x}{1+\cos^4 x}$$

29. Let $y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$ & $z=\sin^{-1}\left(\frac{2x}{1+x^2}\right)$

$$y=\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)=2\tan^{-1}x$$

$$\frac{dy}{dx}=\frac{2}{1+x^2}$$

$$z=\sin^{-1}\left(\frac{2x}{1+x^2}\right)=2\tan^{-1}x$$

$$\frac{dz}{dx}=\frac{2}{1+x^2}$$

$$\therefore \quad \frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}}=1$$

30. Given that $y = x^y$

$\log y = \log x^y = y \log x$

Differentiating both sides, we get

$$\frac{d\log y}{dx}=\frac{dy\log x}{dx}$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=y.\frac{d\log x}{dx}+\log x.\frac{dy}{dx}$$

$$\Rightarrow \left(\frac{1}{y}-\log x\right)\frac{dy}{dx}=\frac{y}{x}$$

$$\Rightarrow \frac{1-y\log x}{y}.\frac{dy}{dx}=\frac{y}{x}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^2}{x(1-y\log x)}$$

31. Given that $2z = x\left(2+\frac{dz}{dx}\right)$

Differentiating both sides, we get

$$2\frac{dz}{dx} = \frac{d}{dx}x\left(2+\frac{dz}{dx}\right)$$

$$\Rightarrow\ 2\frac{dz}{dx} = x\frac{d}{dx}\left(2+\frac{dz}{dx}\right)+\left(2+\frac{dz}{dx}\right).$$

$$\Rightarrow\ 2\frac{dz}{dx} = x.\frac{d^2z}{dx^2}+2+\frac{dz}{dx}$$

$$\Rightarrow\ \frac{dz}{dx} = x.\frac{d^2z}{dx^2}+2$$

Again differentiating w.r.t. x, we get

$$\frac{d^2z}{dx^2} = x\frac{d^3z}{dx^3}+\frac{d^2z}{dx^2}$$

$$\Rightarrow\ x\frac{d^3z}{dx^3}=0$$

$$\Rightarrow\ \frac{d^3z}{dx^3}=0$$

$$\Rightarrow\ \frac{d}{dx}\left(\frac{d^2z}{dx^2}\right)=0$$

$$\Rightarrow\ \frac{d^2z}{dx^2} = \text{A constant.}$$

32. Let x be the side of an equilateral triangle. Let A be its area.

According to the quistion, $\frac{dx}{dt}=2$.

$$A=\frac{\sqrt{3}}{4}x^2$$

$$\frac{dA}{dt}=\frac{\sqrt{3}}{4}\frac{dx^2}{dt}=\frac{\sqrt{3}}{4}.2x.\frac{dx}{dt}$$

$$=\frac{\sqrt{3}}{2}x.2=\sqrt{3}x$$

When x = 10 cm, $\frac{dA}{dt}=\sqrt{3}\times 10=10\sqrt{3}$

$\Rightarrow$ The rate of increase of its area

$= 10\sqrt{3}$ sq unit.

33. Let r be the radius of the sphere.

Volume of the sphere $V=\frac{4}{3}\pi r^3$

Surface area of the sphere $s=4\pi r^3$.

According to the question,

$$\frac{dv}{dr}=2.\frac{ds}{dr}$$

$$\Rightarrow\ \frac{d}{dr}\left(\frac{4}{3}\pi r^3\right)=2\frac{d}{dr}\left(4\pi r^2\right)$$

$$\Rightarrow\ \frac{4}{3}\pi.\ 3r^2=2.4\pi.2r$$

$$\Rightarrow\ 4\pi r^2=16\pi r$$

$\Rightarrow$ r = 4 unit.

34. Let f(x) = sinx – x

$$f'(x)=\cos x-1$$

The function is increasing when $f'(x)>0$

$\Rightarrow$ cos x – 1 > 0

$\Rightarrow$ cos x > 1

$\Rightarrow$ $x\in\phi$.

So there is no set of values of x for which the function is increasing.

35. The given function is

$$f(x)=x^3-12x$$

$$\therefore\ f'(x)=3x^2-12=3\left(x^2-4\right)$$

The function is increasing if $f'(x)\ge 0$

$$\Rightarrow\ 3\left(x^2-4\right)\ge 0$$

$$\Rightarrow\ (x+2)(x-2)>0$$

$\Rightarrow$ x > 2 and x < –2

36. Let $y = \frac{\ln x}{x}, x > 0$

$$\frac{dy}{dx} = \frac{d}{dx}\left(\frac{\ln x}{x}\right)$$

$$= \frac{x.\frac{d\ln x}{dx} - \ln x.\frac{dy}{dx}}{x^2}$$

$$= \frac{x.\frac{1}{x} - \ln x.1}{x^2}$$

$$= \frac{1-\ln x}{x^2}.$$

The function is increasing when $\frac{dy}{dx} > 0$

$\Rightarrow \frac{1-\ln x}{x^2} > 0$

$\Rightarrow 1 - \ln x > 0$

$\Rightarrow \ln x < 1$

$\Rightarrow \ln x < \ln e$

$\Rightarrow x < e$

The function is increasing in (0, e)

37. The equation of the curve is

$\sqrt{x} + \sqrt{y} = \sqrt{a}$ .....(1)

Let P be a point on the curve (1) whose coordinates are (x_1, y_1).

$\therefore \sqrt{x_1} + \sqrt{y_1} = \sqrt{a}$ .....(2)

Differentiating both sides of (1), we get

$$\frac{d\sqrt{x}}{dx} + \frac{d\sqrt{y}}{dx} = \frac{d\sqrt{9}}{dx}$$

$$\Rightarrow \frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

At the point (x_1, y_1) $\frac{dy}{dx} = -\frac{\sqrt{y_1}}{\sqrt{x_1}}$

Equation of the tangent to the curve (1) at the point (1) is $y - y_1 = \frac{dy}{dx}(x - x_1)$

$$\Rightarrow y - y_1 = -\frac{\sqrt{y_1}}{\sqrt{x_1}}(x - x_1)$$

$$\Rightarrow x\sqrt{y_1} + y\sqrt{x_1} = x_1\sqrt{y_1} + y_1\sqrt{x_1}$$

$$\Rightarrow x\sqrt{y_1} + y\sqrt{x_1} = \sqrt{x_1}\sqrt{y_1}\left(\sqrt{x_1} + \sqrt{y_1}\right)$$

$$= \sqrt{x_1}\sqrt{y_1}\sqrt{a}$$

$$\Rightarrow \frac{x}{\sqrt{x_1}\sqrt{a}} + \frac{b}{\sqrt{y_1}\sqrt{a}} = 1$$

Let the tangent at P intersect x-axis at A and y-axis at B.

$\therefore$ $OA = \sqrt{x_1}\sqrt{a}, \quad OB = \sqrt{y_1}\sqrt{a}$.

sum of the intercepts = $\sqrt{x_1}\sqrt{a} + \sqrt{y_1}\sqrt{a}$

$= \sqrt{a}\left(\sqrt{x_1} + \sqrt{y_1}\right)$

$= \sqrt{a}\sqrt{a} = a$ which is constant.

38. Let f(x) = 2 sin x + 3 tan x – 3x

$f'(x)$ = 2 cos x + 3 $\sec^2$x – 3

= 2 cos x + 3 $\tan^2$x > 0 for all $x \in \left(0, \frac{\pi}{2}\right)$

$\Rightarrow$ The function is increasing for all $x \in \left(0, \frac{\pi}{2}\right)$

But f(0) = 2 sin 0 + 3 tan 0 – 3.0

$\Rightarrow f'(0) = 0$

$\therefore f(x) > f(0)$

$\Rightarrow$ 2 sin x + 3 tan x – 3x > 0

$\Rightarrow$ 2 sin x + 3 tan x > 3x for all $x \in \left(0, \frac{\pi}{2}\right)$

39. Given function is

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

$$f'(x) = 12x^3 - 12x^2 - 24x$$

$$= 12x(x^2 - x - 2)$$

$$= 12x(x + 1)(x - 2).$$

When $x < -1$, $f'(x)$ is –ve

When $-1 < x < 0$, $f'(x)$ is +ve

When $0 < x < 2$, $f'(x)$ is –ve

When $x > 2$, $f'(x)$ is +ve

The function is strictly increasing when $-1 < x < 0$ and $x > 2$.

The function is strictly decreasing when $x < -1$ and $0 < x < 2$.

40. The equation of the given curve is

$$x = a\sin^3\theta,\ y = a\cos^3\theta$$

$$\therefore \quad \frac{dx}{d\theta} = 3a\sin^2\theta.\cos\theta$$

$$\frac{dy}{d\theta} = -3a\cos^2\theta.\sin\theta$$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-3a\cos^2\theta\sin\theta}{3a\sin^2\theta\cos\theta} = -\cot\theta$$

At $\theta = \frac{\pi}{4}, \frac{dy}{dx} = -\cot\frac{\pi}{4} = -1$

Slope of the normal = $-\frac{1}{\frac{dy}{dx}} = -\frac{1}{-1} = 1$

When $x = \frac{\pi}{4}$, $x = a\sin^3\frac{\pi}{4} = \frac{a}{2\sqrt{2}}$

$$y = a\cos^3\frac{\pi}{4} = \frac{a}{2\sqrt{2}}$$

The point on the cuve is $\left(\frac{a}{2\sqrt{2}}, \frac{a}{2\sqrt{2}}\right)$

Equation of the normal at this point is

$$y = \frac{a}{2\sqrt{2}} = 1\left(x - \frac{a}{2\sqrt{3}}\right)$$

$\Rightarrow$ $x - y = 0.$

41. The given function is

$$y = x + \frac{1}{x} \qquad ....(1)$$

$$\frac{dy}{dx} = 1 - \frac{1}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}$$

For maximum or minimum, $\frac{dy}{dx} = 0$

$$\Rightarrow \quad 1 - \frac{1}{x^2} = 0$$

$$\Rightarrow \quad x = \pm 1\cdot$$

When x = 1, $\frac{d^2y}{dx^2} = 2$ which is +ve.

The function is minimum at x = 1.

When x = –1, $\frac{d^2y}{dx^2} = -2$ which is –ve.

So the function is maximum at x = –1.

Maximum value of the function = $-1 - \frac{1}{1} = -2$

Minimum value of two function = 2.

42. The given function is

$$f(x) = 4 - x - x^2$$

$$\therefore f'(x) = -1 - 2x$$

$f''(x) = -2$ which on –ve.

The function is maximum when $f'(x) = 0$

$$\Rightarrow -1 - 2x = 0$$

$$\Rightarrow x = -\frac{1}{2}.$$

43. Given function is

$$f(x) = x^4 - 4x^3 - 4x^2 - 1.$$

$$f'(x) = 4x^3 - 12x^2 + 8x$$

$$f''(x) = 12x^2\ 24x + 8$$

For maximum or minimum, $f'(x) = 0$

$\Rightarrow$ $4x^3 - 12x^2 + 8x = 0$

$\Rightarrow$ $4x(x^2 - 3x + 2) = 0$

$\Rightarrow$ $4x(x-1)(x-2) = 0$

$\Rightarrow$ $x = 0, 1, 2.$

When x = 0, $f''(x) = 8$ which is +ve.

The function is minimum at x = 0.

When x = 1, $f''(x) = 12 - 24 + 8 = -4$ which is -ve.

The function is maximum when x = 1,

when x = 2, $f''(x) = 48 - 48 + 8 = 8$ which is +ve.

So the function is minimum when x = 2

44. The given function is

$$f(x) = \sin x \cos x = \frac{1}{2}.2 \sin x \cos x.$$

$$= \frac{1}{2}\sin 2x.$$

$$f'(x) = \cos 2x.$$

$$f''(x) = -2 \sin 2x.$$

For extemum, $f'(x) = 0$

$\Rightarrow$ $\cos 2x = 0$

$\Rightarrow$ $2x = \frac{\pi}{2}$ $\Rightarrow$ $x = \frac{\pi}{4}$.

When $x = \frac{\pi}{4}, f(x) = \frac{1}{2}\sin 2\frac{\pi}{4} = \frac{1}{2}\sin\frac{\pi}{2} = \frac{1}{2}$

Extreme point is $\left(\frac{\pi}{4}, \frac{1}{2}\right)$.

45. Let x and y be the length of two sides of the rectangle.

Let P be its perimeter.

P = 2 (x + y) which is constant

$\Rightarrow$ $x + y = \frac{p}{2}$

$\Rightarrow$ $y = \frac{p}{2} - x$.

Let A be its area.

$$A = xy = x\left(\frac{p}{2} - x\right) = \frac{p}{2}x - x^2$$

$$\frac{dA}{dx} = \frac{p}{2} - 2x$$

$$\frac{d^2A}{dx^2} = -2 \text{ which is negative.}$$

For extremum, $\frac{dA}{dx} = 0$ $\Rightarrow$ $\frac{p}{2} - 2x = 0$

$\Rightarrow$ $x = \frac{p}{4}$

When $x = \frac{p}{4}, \frac{d^2A}{dx^2} = -2$ which is –ve.

So A is maximum when $x = \frac{p}{4}$.

$$y = \frac{p}{2} - \frac{p}{4} = \frac{p}{4}.$$

A is maximum when $x = y = \frac{p}{4}$.

$\Rightarrow$ The rectangle is a square.

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5. Integrals, Application of Integrals, Integrals, Differential Equation

1. Given that $f'(x) = e^x + \frac{1}{1+x^2}$

Integrating both sides, we get

$$\int f'(x)dx = \int e^x dx + \int \frac{1}{1+x^2}dx$$

$\Rightarrow \quad f(x) = e^x + \tan^{-1}x + c \quad$ ....(1)

Given that f(o) = 1.

Putting x = o in (1), we get

$f(0) = e^0 + \tan^{-1} o + c$

$\Rightarrow \quad 1 = 1 + o + c \quad \Rightarrow \quad c = o.$

From, we get $f(x) = e^x + \tan^{-1}x$

2. $\int (\log x)^2 \, dx$.

$= \quad (\log x)^2 . \int dx - \int \left[\frac{d(\log x)^2}{dx} . \int dx \right] dx$

$= \quad x(\log x)^2 - \int \log x \, dx$

$= \quad x(\log x)^2 - [\log x] dx - \left[\left\{ \frac{d\log x}{dx} . \int dx \right\} dx \right]$

$= \quad x(\log x)^2 - x\log x + \int dx$

$= \quad x(\log x)^2 - x\log x + x + c$

3. $\int \frac{2x+9}{(x+3)^2}dx = \int \frac{(2x+6)+3}{(x+3)^2}dx$

$= \quad \int \frac{2(x+3)+3}{(x+3)^2}dx$

$= \quad 2\int \frac{1}{x+3}dx + 3\int \frac{1}{(x+3)^2}dx$

$= \quad 2 \ln (x+3) - \frac{3}{x+3} + C$

4. $I = \int_0^1 \frac{x^5(4-x^2)}{\sqrt{1-x^2}}dx$

$= \quad \int_0^1 x^5 \frac{(3+1-x^2)}{\sqrt{1-x^2}}dx$

$= \quad 3\int_0^1 \frac{x^5}{\sqrt{1-x^2}}dx + \int_0^1 x^5\sqrt{1-x^2}dx \quad$ ....(1)

Let $x = \sin\theta$

$dx = \cos\theta \, d\theta$

When x = 0, $\sin\theta = 0 \Rightarrow \theta = 0$

When x = 1, $\sin\theta = 1 \Rightarrow \theta = \frac{\pi}{2}$

$I = 3\int_0^{\pi/2} \frac{\sin^5\theta}{1-\sin^2\theta}, \cos\theta \, d\theta$

$+ \int_0^{\pi/2} \sin^5\theta\sqrt{1-\sin^2\theta}.\cos\theta \, d\theta$

$= \int_0^{\pi/2} \sin^5\theta \, d\theta + \int_0^{\pi/2} \sin^5\theta \, .\cos\theta \, d\theta$

$= \frac{8}{15} + \frac{8}{105} = \frac{64}{105}$

5. Let $I = \int \frac{\sin\theta\cos\theta}{\sin^2\theta - 2\sin\theta + 3}d\theta$

Let $\sin\theta = t \Rightarrow \cos\theta \, d\theta = dt$

$I = \int \frac{t}{t^2-2t+3}dt$

$= \quad \frac{1}{2} = \int \frac{2t}{t^2-2t+3}dt$

$= \quad \frac{1}{2} = \int \frac{(2t-2)+2}{t^2-2t+3}dt$

$= \quad \frac{1}{2} = \int \frac{2t-2}{t^2-2t+3}dt + \int \frac{1}{t^2-2t+1+2}dt$

$= \quad \frac{1}{2}\ln(t^2-2t+3) + \frac{1}{\sqrt{2}}\tan^{-1}\frac{t-1}{\sqrt{2}} + C$

$= \quad \frac{1}{2}\ln(\sin^2 x - \sin x + 3) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sin x - 1}{\sqrt{2}}\right) + C$

6. Let $I=\int_0^1 x^7\sqrt{\frac{1+x^2}{1-x^2}}dx$

$= \int_0^1 x^7\sqrt{\frac{\left(1+x^2\right)^2}{\left(1-x^2\right)\left(1+x^2\right)}}dx$

$= \int_0^1 x^7\frac{\left(1+x^2\right)}{\sqrt{1-x^4}}dx$

$= \int_0^1 \frac{x^7}{\sqrt{1-x^4}}dx+\int_0^1\frac{x^9}{\sqrt{1-x^4}}dx$ .....(1)

$= I_1 + I_2$ (say)

$I_1 = \int_0^1 \frac{x^7}{\sqrt{1-x^4}}dx$

$= \int_0^1 \frac{x^4.x^3}{\sqrt{1-x^4}}dx$ $x^4=t$, $4x^3dx=dt$

$= \int_0^1 \frac{t.\frac{1}{4}}{\sqrt{1-t}}dt$ $\Rightarrow x^3dx=\frac{1}{4}dt.$

$= \frac{1}{4}\int_0^1\frac{t\ dt}{\sqrt{1-t}}$

$= \frac{1}{4}\int_0^1\frac{\left(1-z^2\right)\times -2\ zd\ z}{z}$ $[1-t=z^2$, $t=1-z^2$

$= -\frac{1}{2}\int_1^0\left(1-z^2\right)dz$

$= \frac{1}{2}\int_0^1\left(1-z^2\right)dz$

$= \frac{1}{2}\left[z-\frac{z^3}{3}\right]_0^1$

$= \frac{1}{2}\left[\left(1-\frac{1}{3}\right)-0\right]$

$= \frac{1}{2}.\frac{2}{3}=\frac{1}{3}$

$I_2 = \int_0^1\frac{x^9}{\sqrt{1-x^4}}dx$

$= \int_0^1\frac{x^8}{\sqrt{1-x^4}}x\,dx$

$= \int_0^1\frac{t^4.\frac{1}{2}dt}{\sqrt{1-t^2}}$ $\left[x^2=t\right.$, $2xdx=dt$

$= \frac{1}{2}\int_0^1\frac{t^4\ dt}{\sqrt{1-t^2}}$ $\left.dxd=\frac{1}{2}dt\right]$

$=\frac{1}{2}\int_0^{\pi/2}\frac{\sin^4\theta}{\sqrt{1-\sin^2\theta}}.\cos\theta\ d\theta$

$= \frac{1}{2}\int_0^{\pi/2}\sin^4\theta\ d\theta$ $t=\sin\theta$, $dt=\cos\theta\ d\theta$

$= \frac{3\pi}{32}$

$I = I_1 + I_2 = \frac{1}{3}+\frac{3\pi}{32}$

7. $\int x^2.\tan^{-1}x\,dx=\int\tan^{-1}x.\ x^2dx$

$= \tan^{-1}x.\int x^2dx-\int\left[\frac{d\tan^{-1}x}{dx}.\int x^2dx\right]dx$

$= \frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\frac{x^3}{1+x^2}dx$

$= \frac{x^3}{3}\tan^{-1}x-\frac{1}{3}\int\frac{x\left(1+x^2\right)-x}{1+x^2}dx$

$= \frac{1}{3}\tan^{-1}x-\frac{1}{3}\int x\,dx+\frac{1}{3}\int\frac{x}{1+x^2}dx$

$= \frac{1}{3}\tan^{-1}x-\frac{x^2}{6}+\frac{1}{6}\ln\left(1+x^2\right)+C$

8. $\int \frac{dx}{x\ln x\sqrt{(\ln x)^2-4}}$

$= \int \frac{1}{\ln x\sqrt{(\ln x)^2-4}}.\frac{1}{x}dx$

$= \int \frac{1}{t\sqrt{t^2-2^2}}dt$ [Where $\ln x = t$, $\frac{1}{x}dx = dt$]

$= \sec^{-1}\frac{t}{2}+C$

$= \sec^{-1}\left(\frac{1}{2}\ln x\right)+C$

9. Let $I = \int \frac{dx}{(1+x)\sqrt{1-x^2}}$

Let $1 + x = \frac{1}{t}$ $\Rightarrow$ $dx = -\frac{1}{t^2}dt$

Also $x = \frac{1}{t} - 1 = \frac{1-t}{t}$

$1-x^2 = 1-\frac{(1-t)^2}{t^2} = \frac{t^2-(1-t)^2}{t^2} = \frac{2t-1}{t^2}$

$\Rightarrow$ $\sqrt{1-x^2} = \frac{\sqrt{2t-1}}{t}$

$I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t}\frac{\sqrt{2t-1}}{t}}$

$= -\int \frac{1}{\sqrt{2t-1}}dt$

Let 2t – 1 = u

$\Rightarrow$ $2dt = du$ $\Rightarrow$ $dt = \frac{1}{2}du$

$I = -\int \frac{1}{\sqrt{u}}.\frac{1}{2}du = -\frac{1}{2}\int u^{-\frac{1}{2}}du$

$= -\sqrt{u}+C = -\sqrt{2t-1}+C$

$= -\sqrt{\frac{1-x}{1+x}}+C.$

10. Let $I = \int_0^1 \frac{\ln x}{\sqrt{1-x^2}}dx$

Let $x = \sin\theta$

$dx = \cos\theta\, d\theta$

when x = 0, $\theta = 0$

when x = 1, $\theta = \frac{\pi}{2}$

$I = \int_0^{\pi/2} \frac{\ln\sin\theta}{\sqrt{1-\sin^2\theta}}.\cos\theta\, d\theta$

$I = \int_0^{\pi/2} \ln\sin\theta\, d\theta.$ ....(1)

or $I = \int_0^{\pi/2} \ln\cos\theta\, d\theta$

$= \int_0^{\pi/2} \ln\cos\theta\, d\theta$ ....(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi/2} \ln\sin\theta\, d\theta + \int_0^{\pi/2} \ln\cos\theta\, d\theta$

$= \int_0^{\pi/2} \ln\sin\theta\cos\theta\, d\theta$

$= \int_0^{\pi/2} \ln\left(\frac{\sin 2\theta}{2}\right)d\theta$

$= \int_0^{\pi/2} \ln\sin\theta\, d\theta - \ln 2\int_0^{\pi/2} d\theta$

$= \int_0^{\pi/2} \ln\sin 2\theta\, d\theta - \frac{\pi}{2}\ln 2$ ......(3)

$\int_0^{\pi/2} \ln\sin 2\theta\, d\theta$

$= \int_0^{\pi} \ln\sin t.\frac{1}{2}dt$ [$2\theta = t$, $2d\theta = dt$]

$= \frac{1}{2}\int_0^{\pi} \ln\sin t\, dt.$

$= \frac{1}{2}\, 2.\int_0^{\pi/2} \ln\sin t\, dt$

$= \int_0^{\pi/2} \ln\sin\theta\, d\theta = I.$

From (3), we get

$$2I = I - \frac{5}{2}\ln 2$$

$$\Rightarrow \quad I = -\frac{\pi}{2}\ln 2.$$

11. Let $I = \int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x} dx$ ....(1)

Also $I = \int_0^{\pi/2} \frac{\sin^n\left(\frac{\pi}{2} - x\right)}{\sin^n\left(\frac{\pi}{2} - x\right) + \cos^n\left(\frac{\pi}{2} - x\right)} dx$

$= \int_0^{\pi/2} \frac{\cos^n x}{\cos^n x + \sin^n x} dx$ ......(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^n x + \cos^n x}{\sin^n x + \cos^n x} dx$$

$$= \int_0^{\pi/2} dx = \frac{\pi}{2}$$

$$\Rightarrow \quad I = \frac{\pi}{4}.$$

12. Let $I = \int \sec x \tan x \sqrt{\tan^2 x - 3}\ dx$

$= \int \sqrt{\sec^2\theta - 4}.\ \sec\theta\ \tan\theta\ d\theta$

Let $\sec\theta = t$

$\Rightarrow \quad \sec\theta \tan\theta\ d\theta = dt$

$\therefore \quad I = \int \sqrt{t^2 - 2^2} dt$

$= \frac{t\sqrt{t^2 - 2^2}}{2} - \frac{2^2}{2}\ln\left(t + \sqrt{t^2 - 2^2}\right) + C$

$= \frac{1}{2}\sec x\sqrt{\sec^2 x - 4} - 2\ln$

$\left(\sec x + \sqrt{\sec^2 x - 4}\right) + C$

13. $I = \int_0^{\pi/4} \frac{1}{\cos x(\cos x + \sin x)} dx$

$= \int_0^{\pi/4} \frac{\sec^2 x}{1 + \tan x} dx$

$= \left[\ln(1 + \tan x)\right]_0^{\pi/4}$

$= \ln\left(1 + \tan\frac{\pi}{4}\right) - \ln(1 + \tan o)$

$= \ln 2$

14. $\int_0^1 \left(\tan^{-1} x + \frac{x}{1 + x^2}\right) dx$

$= \int_0^1 \tan^{-1} x\, dx + \int_0^1 \frac{x}{1 + x^2} dx$

$= \left[\tan^{-1} x.x\right]_0^1 - \int_0^1 \left[\frac{d\tan^{-1}}{dx}.\int dx\right]$

$+ \int_0^1 \frac{x}{1 + x^2} dx$

$= \left[x\tan^{-1} x\right]_0^1 - \int_0^1 \frac{x}{1 + x^2} dx + \int_0^1 \frac{x}{1 + x^2} dx$

$= \frac{\pi}{4}$

15. $\int_0^1 [3x] dx$

$= \int_0^{\frac{1}{3}} [3x] dx + \int_{\frac{1}{3}}^{\frac{2}{3}} [3x] dx + \int_{\frac{2}{3}}^1 [3x] dx$

$= \int_0^{\frac{1}{3}} .o\ dx + \int_{\frac{1}{3}}^{\frac{2}{3}} 1\ dx + \int_{\frac{2}{3}}^1 2\ dx$

$= [x]\int_{\frac{1}{3}}^{\frac{2}{3}} + 2[x]_{\frac{2}{3}}^1 = 1$

16. $\int_0^4 |8 - 3x| dx$

We know $|8 - 3x|$ = 8 – 3x when $8 - 3x \geq 0$

i.e $8 \geq 3x$

i.e $3x \leq 8$

i.e $x \leq \frac{8}{3}$

$= -(8-3x)$ when $8 - 3x < 0$

i.e $8 < 3x$

i.e $3x > 8$

i.e $x > \frac{8}{3}$

$\therefore \int_0^4 |8-3x|\,dx$

$= \int_0^{\frac{8}{3}} |8-3x|\,dx + \int_{\frac{8}{3}}^4 |8-3x|\,dx$

$= \int_0^{\frac{8}{3}} (8-3x)\,dx - \int_{\frac{8}{3}}^4 (8-3x)\,dx$

$= 8$

17. $\int_0^4 \{[x]+|x|\}\,dx$

$= \int_0^4 [x]\,dx + \int_0^4 |x|\,dx$

$= \int_0^1 [x]\,dx + \int_1^2 [x]\,dx + \int_2^3 [x]\,dx + \int_3^4 [x]\,dx + \int_0^4 x\,dx$

$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx + \left[\frac{x^2}{2}\right]_0^4$

$= \int_1^2 dx + 2\int_2^3 dx + 3\int_3^4 dx + \left(\frac{16}{2} - 0\right)$

$= [x]_1^2 + 2[x]_2^3 + 3[x]_3^4 + 8$

$= 2 - 1 + 2(3-2) + 3(4-3) + 8$

$= 1 + 2 + 3 + 8 = 14.$

18. $\int_0^{1.5} [x^2]\,dx$

$= \int_0^1 [x^2]\,dx + \int_1^{\sqrt{2}} [x^2]\,dx + \int_{\sqrt{2}}^{1.5} [x^2]\,dx$

$= \int_0^1 0\,dx + \int_1^{\sqrt{2}} 1.dx + \int_{\sqrt{2}}^{1.5} 2\,dx$

$= 0 + [x]_1^{\sqrt{2}} + 2[x]_{\sqrt{2}}^{1.5}$

$= \sqrt{2} - 1 + 2(1.5 - \sqrt{2})$

$= \sqrt{2} - 1 + 3 - 2\sqrt{2} = 2 - \sqrt{2})$

19. $\int \frac{xe^x}{(1+x)^2}\,dx$

$= \int \frac{(1+x-1)e^x}{(1+x)^2}$

$= \int \frac{1}{1+x}.e^x - \int \frac{1}{(1+x)^2} e^x\,dx$

$= \frac{1}{1+x}\int e^x\,dx - \int \left[\frac{d\left(\frac{1}{1+x}\right)}{dx}.\int e^x dx\right] dx$

$- \int \frac{1}{(1+x)^2} e^x\,dx$

$= \frac{e^x}{1+x} + \int \frac{1}{(1+x)^2} e^x dx - \int \frac{1}{(1+x)^2} e^x dx$

$= \frac{e^x}{1+x} + C$

20. $I = \int \frac{a}{b + c e^x}\,dx = \int \frac{a\,e^{-x}}{b e^{-x} + c}\,dx$ ......(1)

Let $b\,e^{-x} + c = t$

$\Rightarrow -b\,e^{-x}\,dx = dt$

$\Rightarrow e^{-x}dx = -\frac{1}{b}dt$

$I = \int a.\frac{\left(-\frac{1}{b}\right)dt}{t}$

$= -\frac{a}{b}\int \frac{1}{t}\,dt$

$= -\frac{a}{b}\ln t + c$

$= -\frac{a}{b}\ln\left(be^{-x} + c\right) + c$

21. Let I = $\int \frac{e^{x-1}+x^{e-1}}{e^x+x^e}dx$

Let $e^x + x^e = t$

$\therefore \quad \left(e^x + e\,x^{e-1}\right)dx = dt$

$\Rightarrow \quad e\left(e^{x-1} + x^{e-1}\right)dx = dt$

$\Rightarrow \quad \left(e^{x-1} + x^{e-1}\right)dx = \frac{1}{e}dt$

$I = \int \frac{\frac{1}{e}dt}{t} = \frac{1}{e}\int \frac{1}{t}dt$

$= \quad \frac{1}{e}\ln t + c$

$= \quad \frac{1}{e}\ln\left(e^x + x^e\right) + c$.

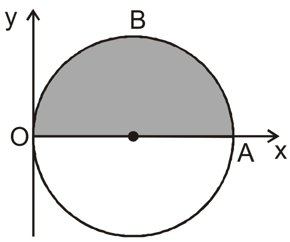
22. The given circle is

$x^2 + y^2 = 2ax$ ......(1)

The centre is (a, 0) and radius = a.

From (1), we get $y^2 = 2ax - x^2$

$\Rightarrow \quad y = \sqrt{2ax = x^2}$ ......(2)



Let x = 2a $\sin^2\theta$

dx = 4a $\sin\theta\cos\theta\, d\theta$

when x = 0, $\theta = 0$

when x = 2a, $\theta = \frac{\pi}{2}$.

Required area = $2\times$ Area of OAB

$= 2\int_0^{2a}\sqrt{2ax - x^2}dx$

$= 2\int_0^{\pi/2}\sqrt{4a^2\sin^2\theta - 4a^2\sin^4\theta}.4a\sin\theta\cos\theta.d\theta$

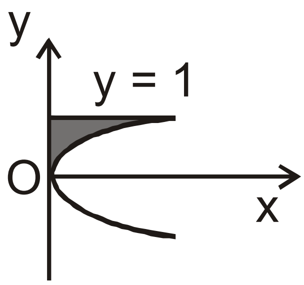
$= 16a^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\, d\theta$

$= 16a^2\frac{\sqrt{\left(\frac{2+1}{2}\right)}\sqrt{\left(\frac{2+1}{2}\right)}}{2\sqrt{\left(\frac{2+2+2}{2}\right)}}$

$= \quad 16a^2\frac{\sqrt{\left(\frac{3}{2}\right)}\sqrt{\left(\frac{3}{2}\right)}}{2\sqrt{(3)}}$

$= \quad \pi a^2$.

23. The equation of the curve is $y^2 = x$ .....(1)
Area is bounded by the curve x = 0, y = 1.



Required area $= \int_0^1 x\, dy$

$= \int_0^1 y^2 dy = \left[\frac{y^3}{3}\right]_0^1$

$= \frac{1}{3}$ sq.unit.

24. The equation of the curve is y = sin x ....(1)
Area is bounded by the curve,

x-axis and x = 0 and $x = \frac{\pi}{2}$.

Required area $= \int_0^{\pi/2} y\, dx = \int_0^{\pi/2}\sin x\, dx$

$= \left[-\cos x\right]_0^{\pi/2}$

$= \left(-\cos\frac{\pi}{2}\right) - \left(-\cos\theta\right) = 1$ sq. unit.

25. The area is bounded by the curve y = 2x,
x-axis and the ordinate x = 3.

Required area $= \int_0^3 y\, dx = \int_0^3 ex\, dx$

$= 2.\left[\frac{x^2}{2}\right]_0^3 = 9$ sq. unit.

26. The area is bounded by y - sin x, x-axis and from x = 0 to $x = \pi$

Required area $= \int_0^{\pi} y\,dx$

$= \int_0^{\pi} \sin x\,dx$

$= \left[-\cos x\right]_0^{\pi}$

$= (-\cos\pi) - (-\cos\theta)$

= -(-1)-(-1)=2 sq. unit.

27. The area is bounded by $y = 3x^2+5$, x-axis and two ordinates x = 1 and x = 2.

Required area $= \int_1^2 y\,dx$

$= \int_1^2 \left(3x^2+5\right)dx$

$= \left[3.\frac{x^3}{3} + 5.x\right]_1^2$

$= \left[x^3+5x\right]_1^2$

$= (8+10)-(1+5)$

= 18–6 = 12 sq. unit

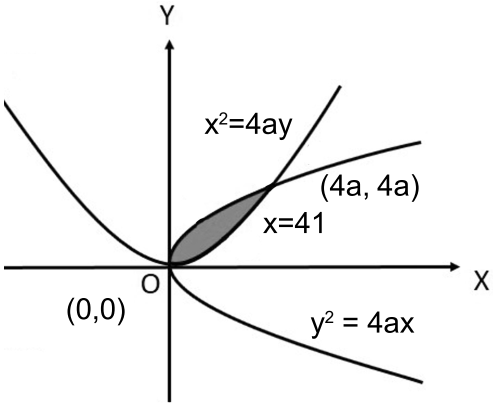
28. The equations of two parabola's are

$y^2 = 4ax$ ..... (1)

$x^2 = 4ay$ ......(2)

The points of intersection of (1) and (2) are (0,0) and (4a, 4a).

Area enclosed between two curves.



$= \int_0^{4a} \sqrt{4ax}\,dx - \int_0^4 \frac{x^2}{4a}dx$

$= 2\sqrt{a}\int_0^{4a} x^{\frac{1}{2}}dx - \frac{1}{4a}\int_0^4 x^2dx$

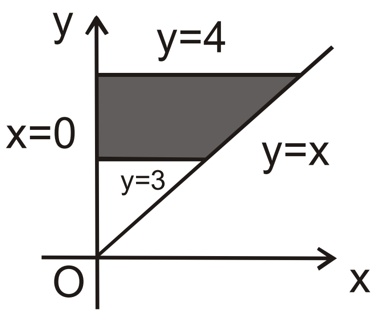
$= \frac{16}{3}a^2$ sq. unit

29. The area is bounded by the curve $y = e^x$, y = 0, x = 2 and x = 4.

The required area $= \int_2^4 y\,dx = \int_2^4 e^x dx$

$= \left[e^x\right]_2^4 = \left(e^4 - e^2\right)$ sq unit.

30. The trapezium is bounded by y = x, x=0, y = 3 and y=4.



Required area

$= \int_3^4 x\,dy = \int_3^4 y\,dy$

$= \left[\frac{y^2}{2}\right]_3^4$

$= \frac{16}{2} - \frac{9}{2} = \frac{7}{2}$ sq. unit.

31. The given differential equation is

$\left(x - \ln y\right)\frac{dy}{dx} = -y\ln y$

$\Rightarrow \quad -y\ln y.\frac{dy}{dx} = x - \ln y$

$\Rightarrow \quad y\ln y.\frac{dy}{dx} + x = \ln y$

$\Rightarrow \quad \frac{dx}{dy} + \frac{1}{y\ln y}x = \frac{1}{y}$

Integrating factor $= e^{\int \frac{1}{y\ln y}dy} = e^{\ln\ln y} = \ln y$.

32. The given differential equation is

$$\frac{dy}{dx}=\frac{y^2}{xy-x^2} \quad ....(1)$$

Let $y = vx$

$$\frac{dy}{dx}=v+x\frac{dv}{dx}.$$

Equation (1) becomes

$$v+x\frac{dv}{dx}=\frac{v^2x^2}{x.vx-x^2}=\frac{v^2}{v-1}$$

$$\Rightarrow \quad x\frac{dv}{dx}=\frac{v^2}{v-1}-v=\frac{v^2-v^2+v}{v-1}=\frac{v}{v-1}$$

$$\Rightarrow \quad \frac{v}{v-1}dv=\frac{dx}{x}$$

Integrating both sides, we get

$$\int\left(1-\frac{1}{V}\right)dv=\int\frac{dx}{x}$$

$$\Rightarrow \quad v-\ln v=\ln x+c$$

$$\Rightarrow \quad \frac{y}{x}-\ln\frac{z}{x}=\ln x+c$$

33. Given differential equation is

$$\frac{dy}{dx}=\frac{1+y^2}{1+x^2}$$

$$\Rightarrow \quad \frac{dy}{1+y^2}=\frac{dy}{1+x^2}$$

Integrating both sides, we get

$$\int\frac{dy}{1+y^2}=\int\frac{dx}{1+x^2}$$

$$\Rightarrow \quad \tan^{-1}y=\tan^{-1}x+c \quad ....(1)$$

when x = 1, then $y=\sqrt{3}$

$\therefore$ From (1), we have $\tan^{-1}\sqrt{3}=\tan^{-1}1+c$

$$\Rightarrow \quad \frac{\pi}{3}=\frac{\pi}{4}+c$$

$$\Rightarrow \quad c=\frac{\pi}{3}-\frac{\pi}{4}=\frac{\pi}{12}$$

The required particular slution is

$$\tan^{-1}y=\tan^{-1}x+\frac{\pi}{12}$$

34. The given differential equation is

$$\left(x+2y^3\right)\frac{dy}{dx}=y$$

$$\Rightarrow \quad y\frac{dx}{dy}=x+2y^3$$

$$\Rightarrow \quad \frac{dx}{dy}-\frac{1}{y}x=2y^2 \quad .....(1)$$

Integrating factor = $e^{\int -\frac{1}{y}dy}=e^{-\ln y}=e^{\ln\frac{1}{y}}=\frac{1}{y}$.

Multiplying both sides of (1) by $\frac{1}{y}$, we get

$$\frac{1}{y}.\frac{dx}{dy}-\frac{1}{y^2}x=2y$$

$$\Rightarrow \quad \frac{d}{dy}\left(\frac{-x}{y}\right)=2y$$

Integrating both sides, $\frac{x}{y}=\int 2y\,dy$

$$\Rightarrow \quad \frac{x}{y}=y^2+c$$

35. Given differential equation is

$$\text{cosec}\,x.\frac{d^2y}{dx^2}=x$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=x\sin x$$

Integrating both sides, we get

$$\frac{dy}{dx}=-x\cos x+\sin x+c$$

$$\Rightarrow \quad dy=\left(-x\cos x+\sin x+c\right)dx$$

INtegrating both sides, we get

$$y = -x\sin x - 2\cos x + cx + d.$$

36. Given differential equation is

$$y\,dy+e^{-y}x\sin x\,dx=0$$

$$\Rightarrow \quad y\,e^y\,dy+\int x\sin x\,dx=10.$$

Integrating both sides, we get

$$\int y\,e^y\,dy+\int x\sin x\,dx=10$$

$$y\int e^y dy - \int\left[\frac{dy}{dy}\int e^y dy\right]dy$$

$$+\ x\int \sin x\,dx - \int\left[\frac{dx}{dx}\int \sin x\,dx\right]dx = c$$

$$\Rightarrow\ xe^y - \int e^y dy + x(-\cos x) - \int(-\cos x)dx = c$$

$$\Rightarrow\ xe^y - e^y - x\cos x + \sin x = c$$

37. The given differential equation is

$$\left(x^2-1\right)\frac{dy}{dx} + 2xy = 1$$

$$\Rightarrow\ \frac{dy}{dx} + \frac{2x}{x^2-1}y = \frac{1}{x^2-1} \quad .....(1)$$

Integrating factor = $e^{\int \frac{2x}{x^2-1}dx} = e^{\ln(x^2-1)} = x^2-1$

Multiplying both sides of (1) by x^2-1 we get

$$\left(x^2-1\right)\frac{dy}{dx} + 2xy = 1$$

$$\Rightarrow\ \frac{d}{dx}\left[y\left(x^2-1\right)\right] = 1$$

Integrating both sides, we get

$$y\left(x^2-1\right) = \int dx$$

$$\Rightarrow\ y\left(x^2-1\right) = x + c$$

38. Given differential equation is

$$x^2(y-1)dx + y^2(x-1)dy = 0$$

$$\Rightarrow\ \frac{x^2}{x-1}dx + \frac{y^2}{y-1}dy = 0$$

Integrating both sides, we get

$$\int\frac{x^2}{x-1}dx + \int\frac{y^2}{y-1}dy - \int 0$$

$$\Rightarrow\ \int\frac{\left(x^2-1\right)+1}{x-1}dx + \int\frac{\left(y^2-1\right)+1}{y-1}dy = c$$

$$\Rightarrow\ \int\left(x+1+\frac{1}{x-1}\right)dx + \int\left(y+1+\frac{1}{y-1}\right)dy = c$$

$$\Rightarrow\ \frac{x^2}{2} + x + \ln(x-1) + \frac{y^2}{2} + y + \ln(y-1) = c$$

39. The given differential equation is $\frac{d^2y}{dx^2} = 6x$

Integrating both sides, we get

$$\frac{dy}{dx} = 3x^2 + c \quad ....(1)$$

When x = 0, $\frac{dy}{dx} = 2$

From (1), we get 2 = 3.0 + c $\Rightarrow$ c = 2

Equation (1) be comes

$$\frac{dy}{dx} = 3x^2 + 2$$

$$\Rightarrow\ dy = \left(3x^2+2\right)dx$$

$$\therefore\ \int dy = \int\left(3x^2+2\right)dx$$

$$\Rightarrow\ y = x^3 + 2x + D \quad ....(2)$$

When x = 0, y = 1.

From (2), we get 1 = 0 + 0 + D $\Rightarrow$ D = 1

From (2), we get $y = x^3 + 2x + 1$ which is the required particular solution.

40. The given differential equation is

$$\left(1+y^2\right)dx + \left(x - e^{-\tan^{-1}y}\right)dy = 0$$

$$\Rightarrow\ \left(1+y^2\right)\frac{dx}{dy} + x = e^{-\tan^{-1}y}$$

$$\Rightarrow\ \frac{dx}{dy} + \frac{1}{1+y^2}x = e^{-\tan^{-1}y}$$

Integrating factor = $e^{\int\frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$.

41. Given differential equation is

$$\frac{dy}{dx} = \frac{1}{x^2-7x+12} = \frac{1}{(x-3)(x-4)} \quad ....(1)$$

Let $\frac{1}{(x-3)(x-4)} = \frac{A}{x-3} + \frac{B}{x-4}$ ....(2)

$\Rightarrow$ 1 = A (x – 4) + B (x – 3)

When x = 3, we get 1 = A(–1) $\Rightarrow$ A= –1

When x = 4, we get 1 = A.0 + B $\Rightarrow$ B = 1.

From (2), we get

$$\frac{1}{(x-3)(x-4)} = \frac{-1}{x-3} + \frac{1}{x-4}.$$

From (1), we get

$$\frac{dy}{dx}=\frac{1}{x-4}-\frac{1}{x-3}$$

$$\Rightarrow \quad dy=\left(\frac{1}{x-4}-\frac{1}{x-3}\right)dx$$

Integrating both sides, we get

$$\int dy=\int\left(\frac{1}{x-4}-\frac{1}{x-3}\right)dx$$

$$\Rightarrow \quad y=\ln(x-4)-\ln(x-3)+C$$

42. The solution is

$$y = at + be^t \qquad .....(1)$$

$$\therefore \quad \frac{dy}{dt}=a+be^t \qquad .....(2)$$

$$\frac{d^2y}{dt^2}=be^t \qquad ......(3)$$

From (2) and (3), we get

$$\frac{dy}{dt}=a+\frac{d^2y}{dt^2}$$

$$\Rightarrow \quad a=\frac{dy}{dt}-\frac{d^2y}{dt^2} \qquad ....(4)$$

From (1), (3) and (4), we get

$$y=\left(\frac{dy}{dt}-\frac{d^2y}{dt^2}\right)t+\frac{d^2y}{dt^2}$$

Which is the required differential equation.

43. The general solution is

$$ax^2 + by = 1 \qquad .....(1)$$

Differentiating both sides w.r.t x, we get

$$2ax+b\frac{dy}{dx}=0 \qquad .....(2)$$

Again differentiating w.r.t x, we get

$$2a+b\frac{d^2y}{dx^2}=0$$

$$\Rightarrow \quad 2a=-b\frac{d^2y}{dx^2} \qquad ....(3)$$

From (2) and (3), we get

$$-b\frac{d^2y}{dx^2}.x+b\frac{dy}{dx}=0$$

$$\Rightarrow \quad -x\frac{d^2y}{dx^2}+\frac{dy}{dx}=0$$

44. Given equation of the family of curves is

$$(x-a)^2 + 2y^2 = a^2 \qquad ....(1)$$

Differentiating both sides w.r.t x, we get

$$2(x-a)+4y\frac{dy}{dx}=0$$

$$\Rightarrow \quad x-a+2y\frac{dy}{dx}=0$$

$$\Rightarrow \quad a=x+2y\frac{dy}{dx}$$

Putting the value of a in (1), we get

$$\left(x-x-2y\frac{dy}{dx}\right)^2+2y^2=\left(x+2y\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad 4y^2\left(\frac{dy}{dx}\right)^2+2y^2=\left(x+2y\frac{dy}{dx}\right)^2$$

$$\Rightarrow \quad y^2=2y\frac{dy}{dx}+2y^2\left(\frac{dy}{dx}\right)^2$$

45. $y=Ae^{2x}+Be^{-2x}$

$$\therefore \quad \frac{dy}{dx}=2Ae^{2x}-2Be^{-2x}$$

$$\frac{d^2y}{dx^2}=4Ae^{2x}-2Be^{-2x}$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=4\left(Ae^{2x}-2Be^{-2x}\right)$$

$$\Rightarrow \quad \frac{d^2y}{dx^2}=4y.$$

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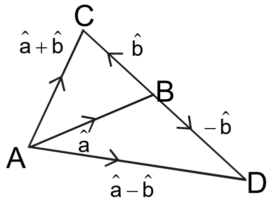
6. Vectors, Three dimensional Geometry

1. Let $\overrightarrow{AB} = \hat{a}$, $\overrightarrow{BC} = \hat{b}$

$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \hat{a} + \hat{b}$

Given that $|\overrightarrow{AB}| = |\hat{a}| = 1$



$|\overrightarrow{BC}| = |\hat{b}| = 1$

$|\overrightarrow{AC}| = |\vec{a} + \vec{b}| = 1$

ABC is an equilateral triangle.

Let us produce CB to D such that BC = BD

$\therefore \quad \overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = \hat{a} + (-\hat{b}) = \hat{a} - \hat{b} \cdot$

$\angle CAD = 60^0 + 30^0 = 90^0$

$\angle ACD = 60^0$

From the $\Delta\, ACD$, $\tan \angle ACD = \dfrac{|\overrightarrow{AD}|}{|\overrightarrow{AC}|}$

$\Rightarrow \quad \tan 60^0 = \dfrac{|\hat{a} - \hat{b}|}{|\hat{a} + \hat{b}|}$

$\Rightarrow \quad \sqrt{3} = \dfrac{|\hat{a} - \hat{b}|}{1}$

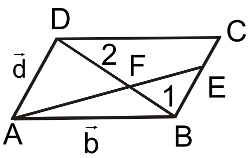
$\Rightarrow \quad |\hat{a} - \hat{b}| = \sqrt{3}$

2. ABCD is parallelogram

Let $\overrightarrow{AB} = \vec{b}, \overrightarrow{AD} = \vec{d}$.

Let E be the middle point of BC.

$\therefore \quad \overrightarrow{BE} = \dfrac{\vec{d}}{2}$



$\overrightarrow{AE} = \overrightarrow{AB} + \overrightarrow{BE} = \vec{b} + \dfrac{\vec{d}}{2} = \dfrac{1}{2}(2\vec{b} + \vec{d})$

Let F be a point on BD which divides it is the ratio 1:2.

$\overrightarrow{AF} = \dfrac{2\vec{b} + 1.\vec{d}}{1+1} = \dfrac{1}{3}(2\vec{b} + \vec{d}) = \dfrac{2}{3}\left(\dfrac{2\vec{b} + \vec{d}}{2}\right)$

$= \dfrac{2}{3}\overrightarrow{AE}$.

Thus AE dividing BD is the ratio 1 : 2.

3. Let $\vec{a} = (\lambda + 1)\hat{i} + 2\hat{j} + \hat{k}$

$\vec{b} = -\hat{c} + \lambda\hat{j} + \hat{k}$

$\vec{c} = \lambda\hat{i} + \hat{j} + 3\hat{k}$

If three vectors are coplanar, then

$$\begin{vmatrix} \lambda+1 & 2 & 1 \\ -1 & \lambda & 1 \\ \lambda & 1 & 3 \end{vmatrix} = 0$$

$\Rightarrow \quad (\lambda + 1)(3\lambda - 1) - 2(-3 - \lambda) + 1(-1 - \lambda^2) = 0$

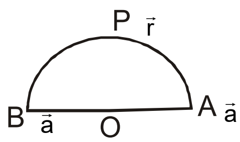
$\Rightarrow \quad \lambda^2 + 2\lambda + 2 = 0$

$\Rightarrow \quad \lambda = \dfrac{-2 \pm 2i}{2} = -1 \pm i$ which is imaginary.

So the vcectors are not coplaner for any real values of λ

4. Given that $\vec{a} = (2,-2,1) = 2\hat{i} - 2\hat{j} + \hat{k}$

$\vec{b} = (2,3,6) = 2\hat{i} + 3\hat{j} + 6\hat{k}$



$\vec{c} = (-1,0,2) = -\hat{i} + 0\hat{j} + 2\hat{k}$.

$\vec{a} + \vec{b} - \vec{c} = 2\hat{i} - 2\hat{j} + \hat{k} + 2\hat{i} + 3\hat{j} + 6\hat{k} - (\hat{i} + 0\hat{j} + 2\hat{k})$

$= 5\hat{i} + \hat{j} + 5\hat{k}$

$|\vec{a} + \vec{b} - \vec{c}| = \sqrt{5^2 + 1^2 + 5^2} = \sqrt{25 + 1 + 25} = \sqrt{51}$,

The d.cs of the vector are $\left\langle \dfrac{5}{\sqrt{5}}, \dfrac{1}{\sqrt{5}}, \dfrac{5}{\sqrt{5}}, \right\rangle$

5. Let APB be a semicircle and O be the centre of the circle.

Let $\overrightarrow{OA} = \vec{a}$, $\overrightarrow{OB} = -\vec{a}$

and let $\overrightarrow{OP} = \vec{r}$

$\overrightarrow{AP} = \vec{r} - \vec{a}$

$\overrightarrow{BP} = \vec{r} + \vec{a}$

$\overrightarrow{AP}.\overrightarrow{BP} = (\vec{r} - \vec{a}).(\vec{a} + \vec{a})$

$= \vec{r}^2 - \vec{a}^2$

$= |\vec{r}|^2 - |\vec{a}|^2 = 0 \quad (\because |\vec{r}| = |\vec{a}|)$

$\Rightarrow$ AP is perpendicular to BP

$\Rightarrow$ $\angle APB$ is right angle.

6. $\vec{a}, \vec{b}, \vec{c}$ are three multiply perpendicular vectors.

$\vec{b} \times \vec{c}$ is perpendicular to $\vec{b}$ and $\vec{c}$.

Also $\vec{a}$ is perpendicular to $\vec{b}$ and $\vec{c}$

$\therefore$ $\vec{a}$ and $\vec{b} \times \vec{c}$ are parallel

$\therefore$ $\vec{a}.(\vec{b} \times \vec{c}) = |\vec{a}||\vec{b} \times \vec{c}|$

$= |\vec{a}||\vec{b}||\vec{c}|$ and 0

$= |\vec{a}||\vec{b}||\vec{c}|$

$\therefore$ $\left[\vec{a}.(\vec{b} \times \vec{c})\right]^2 = |\vec{a}|^2 |b|^2 |\vec{c}|^2 = a^2b^2c^2$

7. $\left[\vec{a} + \vec{b} \;\; \vec{b} + \vec{c} \;\; \vec{c} + \vec{a}\right]$

$= (\vec{a} + \vec{b}).\left[(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\right]$

$= (\vec{a} + \vec{b}).\left[\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + \vec{c} \times \vec{c} + \vec{c} \times \vec{a}\right]$

$= (\vec{a} + \vec{b}).\left[\vec{b} \times \vec{c} + \vec{b} \times \vec{a} + 0 + \vec{c} \times \vec{a}\right]$

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{a}.(\vec{b} \times \vec{a}) + \vec{a}.(\vec{c} \times \vec{a})$

$+ \vec{b}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{b} \times \vec{a}) + \vec{b}.(\vec{c} \times \vec{a})$

$= \vec{a}.(\vec{b} \times \vec{c}) + \vec{b}.(\vec{c} \times \vec{a})$

$= \left[\vec{a}\vec{b}\vec{c}\right] + \left[\vec{a}\;\vec{b}\;\vec{c}\;\right] = 2\left[\vec{a}\;\vec{b}\;\vec{c}\right]$

8. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 2\hat{i} - \alpha\hat{j} + 3\hat{k}$.

Two vectors are orthogonal.

$\therefore$ $\vec{a}, \vec{b} = 0$.

$(\hat{i} + \hat{j} + \hat{k}).(2\hat{i} - \alpha\hat{j} + \hat{k}) = 0$

$\Rightarrow$ $2 - \alpha + 3 = 0$

$\Rightarrow$ $\alpha = 5$.

9. $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{b} = 2\hat{i} + \hat{j} - \hat{k}$

$\vec{c} = 0\hat{i} + \hat{j} + \hat{k}$.

$$\left[\vec{a}\;\vec{b}\;\vec{c}\right] = \begin{vmatrix} 1 & -2 & 3 \\ 2 & 1 & -1 \\ 0 & 1 & 1 \end{vmatrix}$$

= 1 (1 + 1) – (-2) (2 – 0) + 3 (2 – 0)

= 2 + 4 + 6 = 12

10. Given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$

$\vec{c} = 0\hat{i} + \hat{j} - \hat{k}$

Let $\vec{b} = x\hat{i} + y\hat{j} - z\hat{k}$

Given that $\vec{a}.\vec{b} = 3$

$\Rightarrow$ $(\hat{i} + \hat{j} + \hat{k}).(x\hat{i} + y\hat{j} + z\hat{k}) = 3$

$\Rightarrow$ x + y + z = 3

$\Rightarrow$ $\vec{a} \times \vec{b} = \vec{c}$

Again $\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ x & y & z \end{vmatrix} = \hat{j} - \hat{k}$

$\Rightarrow$ $\hat{i}(x - y) - \hat{j}(y - x) + \hat{k}(y - x) = \hat{j} - \hat{k}$

$\Rightarrow$ z – y = 0, –(y–x)=1, y – x = –1

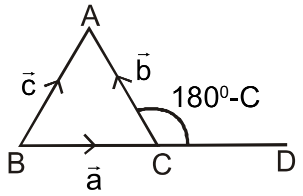
$\Rightarrow$ z – y = 0 x - y = 1, x –y = 1 ...(2)

$x = \frac{5}{3}, y = \frac{2}{3}, z = \frac{2}{3}$

$\therefore$ $\vec{b} = \frac{5}{3}\hat{i} + \frac{2}{3}\hat{j} + \frac{2}{3}\hat{k}$.

11. ABC is a triangle.

Let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$,



$\overrightarrow{AB} = \vec{c}$

We know

$\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$

$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{c} = -\left(\vec{a} + \vec{b}\right)$

$\therefore \quad \vec{c}.\vec{c} = -\left(\vec{a} + \vec{b}\right).-\left(\vec{a} + \vec{b}\right)$

$\vec{c}^2 = \left(\vec{a} + \vec{b}\right).\left(\vec{a} + \vec{b}\right)$

$= \vec{a}^2 + \vec{b}^2 + 2\vec{a}.\vec{b}$

$\Rightarrow \quad \left|\vec{c}\right|^2 = a^2 + b^2 - 2ab\,\cos c$

12. Let $\vec{a} = 2\hat{i} + 2\hat{j} = 2\hat{i} + 2\hat{j} + 0\hat{k}$

$\vec{b} = \hat{i} - \hat{k} = \hat{i} + 0\hat{j} - \hat{k}$

vector area of the parallelogram = $\vec{a} \times \vec{b}$

$$= \begin{vmatrix} \hat{i} & \hat{i} & \hat{k} \\ 2 & 2 & 0 \\ 1 & 0 & -1 \end{vmatrix}$$

$= -2\hat{i} + 2\hat{j} - 2\hat{k}$

Area of the parallelogram = $[-2\hat{i} + 3\hat{j} + 2]$

$= \sqrt{(-2)^2 + 2^2 + (-2)^2} = \sqrt{4+4+4}$

$= \sqrt{12}$ sq. unit

13. Let 0 be the origin.

The position vector of A, B, C are $2\hat{i} + \hat{j} - \hat{k}$

$3\hat{i} - 2\hat{j} + \hat{k}$ and $\hat{i} + 4\hat{j} - 3\hat{k}$

$\therefore \quad \overrightarrow{OA} = 2\hat{i} + \hat{j} - \hat{k}$

$\overrightarrow{OB} = 3\hat{i} - 2\hat{j} + \hat{k}$

$\overrightarrow{OC} = \hat{i} + 4\hat{j} - 3\hat{k}$

$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = (3\hat{i} - 2\hat{j} + \hat{k}) - (2\hat{i} + \hat{j} - \hat{k})$

$= \hat{i} - 3\hat{j} + 2\hat{k}$

$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$

$= \left(\hat{i} + 4\hat{j} - 3\hat{k}\right) - \left(2\hat{i} + \hat{j} - \hat{k}\right)$

$= -\hat{i} + 3\hat{j} - 2\hat{k}$

$= -\left(\hat{i} - 3\hat{j} + 2\hat{k}\right) = -\overrightarrow{AB}$

Thus three points A, B & C are collinear.

14. Given that $\vec{a} = 3\hat{i} + 6\hat{j} + 9\hat{k}$

$\vec{b} = 2\hat{i} + 2\hat{j} - \hat{k}$

Let θ be the angle between them.

$$\cos\theta = \frac{\vec{a}.\vec{b}}{\left|\vec{a}\right|\left|\vec{b}\right|}$$

$$= \frac{\left(3\hat{i} + 6\hat{j} + 9\hat{k}\right).\left(2\hat{i} + 2\hat{j} - \hat{k}\right)}{\sqrt{9+36+81}\sqrt{4+4+1}}$$

$$= \frac{3}{\sqrt{126}}.$$

The scalar projection of $\vec{a}$ on $\vec{b}$

$= \left|\vec{a}\right|\cos\theta$

$= \sqrt{126}.\dfrac{3}{\sqrt{126}} = 3$

15. Let $\vec{a} = -\hat{i} + \lambda\hat{j} - \lambda\hat{k}$

$\vec{b} = 2\hat{i} + 4\hat{j} - 5\hat{k}$

$\vec{c} = -2\hat{i} + 4\hat{j} - 4\hat{k}$

The vectors are coplanar

$\therefore \quad \vec{a}.\left(\vec{b} \times \vec{c}\right) = 0$

$$\Rightarrow \begin{vmatrix} -1 & \lambda & -\lambda \\ 2 & 4 & 5 \\ -2 & 4 & -4 \end{vmatrix} = 0$$

$\Rightarrow -1(-16-20) - \lambda(-8+10) + (-\lambda)(8+8) = 0$

$\Rightarrow \lambda = 2$

16. Given that $|\vec{a}| = 3, |\vec{b}| = 5$ and $|\vec{c}| = 7$

Also $\vec{a} + \vec{b} + \vec{c} = 0$

$\Rightarrow \quad \vec{a} + \vec{b} = -\vec{c}$

$\therefore \quad (\vec{a} + \vec{b}).(\vec{a} + \vec{b}) = (-\vec{c}).(-\vec{c})$

$\Rightarrow \quad \vec{a}^2 + \vec{b}^2 + 2\vec{a}.\vec{b} = \vec{c}^2$

$\Rightarrow \quad |\vec{a}|^2 + |\vec{b}|^2 + 2\vec{a}.\vec{b} = |\vec{c}|^2$

$\Rightarrow \quad \vec{a}.\vec{b} = \dfrac{|\vec{c}|^2 - |\vec{a}|^2 - |\vec{b}|^2}{2}$

$= \dfrac{7^2 - 3^2 - 5^2}{2}$

$= \dfrac{49 - 9^2 - 25}{2}$

$\Rightarrow \quad |\vec{a}||\vec{b}|\cos\theta = \dfrac{15}{2}$

$\Rightarrow \quad 3.5.\cos\theta = \dfrac{15}{2}$

$\Rightarrow \quad \cos\theta = -\dfrac{1}{2} = \cos\dfrac{\pi}{3}$

$\Rightarrow \quad \theta = \dfrac{\pi}{3}.$

17. Let $\overrightarrow{OA} = \hat{a}, \overrightarrow{AB} = \hat{b}$

$\overrightarrow{OB} = \hat{a} + \hat{b}\cdot$

Let us produce BA to C

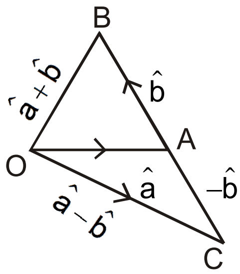
such that AC = AB

$\therefore \quad \overrightarrow{AC} = -\hat{b}$

$\overrightarrow{OC} = \hat{a} - \hat{b}$

Given that $|\hat{a} - \hat{b}| = \sqrt{3}$

$\Rightarrow \quad |\hat{a} - \hat{b}|^2 = 3$



$\Rightarrow \quad (\hat{a} - \hat{b})^2 = 3$

$\Rightarrow \quad \hat{a}^2 + \hat{b}^2 - 2\hat{a}.\hat{b} = 3$

$\Rightarrow \quad 1 + 1 - 2\hat{a}.\hat{b} = 3$

$\Rightarrow \quad 2\hat{a}.\hat{b} = -1$

$\Rightarrow \quad \hat{a}.\hat{b} = -\dfrac{1}{2}$ = cos 120^0

$\Rightarrow \quad \angle OAC = 120^0$

$\Rightarrow \quad \angle OAC = 60^0$

18. Since $|\overline{OA}| = |\overline{AB}| = 1.$

$\therefore$ Given that $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = 4\hat{i} - 2\hat{j} + 3\hat{k}$

and $\vec{c} = \hat{i} - 2\hat{j} + \hat{k}$

$2\vec{a} - \vec{b} + 3\vec{c} = 2(\hat{i} + \hat{j} + \hat{k})$

$-(4\hat{i} - 2\hat{j} + 3\hat{k}) + 3(\hat{i} - 2\hat{j} + \hat{k})$

$= \hat{i} - 2\hat{j} + 2\hat{k}$

$|2\vec{a} - \vec{b} + 3\vec{c}| = \sqrt{1^2 + (-2)^2 + 2^2} = 3$

Unit vector along $2\vec{a} - \vec{b} + 3\vec{c}$

$= \dfrac{2\vec{a} - \vec{b} + 3\vec{c}}{|2\vec{a} - \vec{b} + 3\vec{c}|} = \dfrac{1}{3}(\hat{i} - 2\hat{j} + 2\hat{k})$

$= \left(\dfrac{1}{3}\hat{i} - \dfrac{2}{3}\hat{j} + \dfrac{2}{3}\hat{k}\right).$

Vector of magnitude 6 unit and parallel to the vector $2\vec{a} - \vec{b} + 3\vec{c} \quad = \quad 6.\left(\dfrac{1}{3}\hat{i} - \dfrac{2}{3}\hat{j} + \dfrac{2}{3}\hat{k}\right)$

$= \quad 2\hat{i} - 4\hat{j} + 4\hat{k}$

19. Here A, B, C and D are four points whose P.Vs are $\hat{i} + 2\hat{j} - 3\hat{k}$, $2\hat{i} - \hat{j} - \hat{k}$, $-\hat{i} + 2\hat{j} + 2\hat{k}$ and $2\hat{j} + 2\hat{k}$ respectively.

$\overrightarrow{AB}$ = P. V. of B – P.V. of A = $\hat{i} - 2\hat{j} - 2\hat{k}$

$\overline{AC}$ = P.V. of C – P.V. of A

$= -2\hat{i}+\hat{j}-\hat{k}$

$\overline{AD}$ = P.V. of D – P.V. of A

$= -\hat{i}+\hat{j}-\hat{k}$

$$\overline{AB}.\left(\overline{AC}\times\overline{AD}\right) = \begin{vmatrix} 1 & -2 & -2 \\ -2 & 1 & -1 \\ -1 & 1 & -1 \end{vmatrix}$$

= 1 (–1 + 1) – (–2) (2 – 1) + (–2) (–2 + 1)

= 0

So four points are coplanar.

20. The sides of a parallelopiped are $-\hat{j}, \hat{k}$ and $-\hat{i}$

It volume $= -\hat{j}.\left(\hat{k}\times-\hat{c}\right)$

$= -\hat{j}.-\hat{j}=1$ cubic unit.

21. First line is

$\vec{r}=2\hat{i}-5\hat{j}+\hat{k}+\lambda\left(3\hat{i}+2\hat{j}+6\hat{k}\right)$

$\vec{a}_1=2\hat{i}-5\hat{j}+\hat{k}$

$\vec{b}_1=3\hat{i}+2\hat{j}+6\hat{k}$

Second line is

$\vec{r}=7\hat{i}-6\hat{j}-6\hat{k}+\mu\left(\hat{i}+2\hat{j}+2\hat{k}\right)$

$\vec{a}_2=7\hat{i}-6\hat{j}-6\hat{k}$

$\vec{b}_2=\hat{i}+2\hat{j}+2\hat{k}$

Let θ be the angle between the lines (1) and (2).

$$\cos\theta=\frac{\vec{b}_1.\vec{b}_2}{\left|\vec{b}_1\right|\left|\vec{b}_2\right|}$$

$$= \frac{\left(3\hat{i}+2\hat{j}+6\hat{k}\right).\left(\hat{i}+2\hat{j}+2\hat{k}\right)}{\sqrt{9+4+36}\sqrt{1+4+4}}$$

$$= \frac{3.1+2.2+6.2}{\sqrt{49}\ \sqrt{9}} = \frac{19}{21}$$

$$= \theta=\cos^{-1}\left(\frac{19}{21}\right)$$

22. Two given lines are

$$\frac{x+1}{7}=\frac{y+1}{-1}=\frac{z+1}{1} \quad .....(1)$$

and $$\frac{x-3}{1}=\frac{y-5}{-2}=\frac{2-7}{1} \quad ......(2)$$

The shortest distance

$$\frac{\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}}{\left(b_1c_2-b_2c_1\right)^2+\left(c_1a_2-c_2a_1\right)^2+\left(a_1b_2-a_2b_1\right)^2}$$

$$= \frac{\begin{vmatrix} -4 & -6 & -8 \\ 1 & -2 & 1 \\ 7 & -6 & 1 \end{vmatrix}}{\sqrt{\left(-2+6\right)^2+\left(7-1\right)^2+\left(-6+14\right)^2}}$$

$= \sqrt{116}$

23. The given point is (–1, 3, 2).

The equation of the plane passing through (–1, 3, 2) is

a(x+1) + b (y–3) + c(z–2) = 0 ......(1)

Given planes are

x + 2y + 2z = 5 ....(2)

3x + 3y + 2z = 5 ....(3)

Since the plane (1) is perpendicular to the planes (2) & (3), we have

$a+2b+2c=0$

$3a+3b+2c=0$

By cross multiplications, we get

$$\frac{a}{2.2-3.2}=\frac{b}{2.3-1.2}=\frac{c}{1.2-3.2}$$

$$\Rightarrow \frac{a}{-2}=\frac{b}{4}=\frac{c}{-3}=k \text{ (say)}$$

$\Rightarrow$ a = –2k, b = 4k, c = – 3k.

From (1), the repuired plane is

– 2K (x + 1) + 4k (y – 3) – 3K (z–2)=0

$\Rightarrow$ –2 (x+1) + 4 (y – 3) – 3 (z - 2) = 0

$\Rightarrow$ 2x – 4y + 3z + 8 = 0

24. The first line is

$$\frac{x+4}{3}=\frac{y+6}{5}=\frac{z-1}{-2} \quad ....(1)$$

The 2nd line is

$3x + 2y + z + 5 = 0 = 2x + 3y + 4z - 4$

The symmetric form of the above line is

$$\frac{x-\left(-\frac{7}{13}\right)}{-11}=\frac{y-\frac{23}{13}}{-10}=\frac{z=0}{13} \quad ......(2)$$

The lines are coplaner if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Here $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$

$$=\begin{vmatrix} -\frac{7}{13}+4 & \frac{22}{13}+6 & 0-1 \\ 3 & 5 & -2 \\ -11 & -10 & 13 \end{vmatrix}\neq 0$$

So two lines are not coplanar.

25. Equation of the plane passing through (1, 2, 3)

$a (x - 1) + b (y-2) + c(z+3)=0$ ......(1)

Since it is passing through (2, 3, -4),

we have $a(2-1) + b (3-2) + c (-4 + 3) = 0$

$\Rightarrow$ $a + b - c = 0$ ......(2)

Since the plane (1) is perpendicular to the

$x + y + z + 1 = 0$, we have

$a + b + c = 0$ ......(3)

From (2) & (3) by cross multiplication, we get

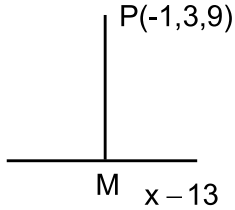
$$\frac{9}{-1}=\frac{b}{0}=\frac{c}{1}$$

The required plane is

$-1 (x - 1) + 0 (y-2) + 1(2+3) = 0$

$\Rightarrow$ $x - 2 - 4 = 0$

26. Let P be the given point whose coordinates are (–1, 3, 9).



$$\frac{x-13}{5}=\frac{y+8}{-8}=\frac{z-31}{1}$$

The given line is

$$\frac{x-13}{5}=\frac{y+8}{-8}=\frac{z-31}{1}=r \text{ (say).} \quad .....(1)$$

Let PM be the perpendicular from P to the line (1)

Any point on the line (1) is (5r+13, –8r–8, r+3) for some value of r.

The d.rs of PM are

$$\langle 5r+13-(-1),\ -8r-8-3,\ r+31-9\rangle$$

$$= \quad \langle 5r+14,\ -8r-11,\ r+22\rangle$$

Since PM is perpendicular to the line (1), we have

$5 (5r+14)-8(-8r-11)+1(r+22)=0$

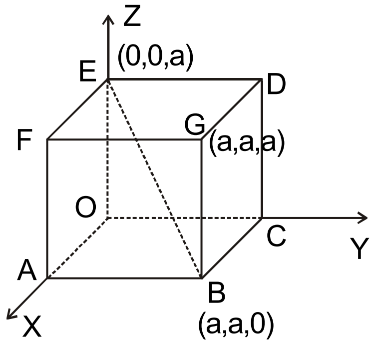
$\Rightarrow$ $r = -2$

The coordinates of M are (3, 8, 29)

$$PM=\sqrt{\left[3-(-1)^2\right]+(8-3)^2+(29-9)^2}$$

$$=\sqrt{441}=21$$

27. Let OABCDEFG be a cube, the length of each edge is ‘a’ each.



Let O be the origin.

Let OA along x-axis, OC along is y-axis and OZ along z-axis.

The coordinates of the corner points area 0(0,0,0) A(a,0,a) & G (a, a, a).

The d.rs of EB are $\langle a\text{-}0,\ a\text{-}0,\ o\text{-}a\rangle=\langle a,a,-a\rangle$

Let θ be the angle between OG & EB.

$$\cos\theta = \frac{a.a + a.a + (-a).a}{\sqrt{a^2 + a^2 + a^2}\sqrt{a^2 + a^2 + (-a)^2}}$$

$$= \frac{a^2 + a^2 - a^2}{\sqrt{3}a.\sqrt{3}a} = \frac{1}{3}$$

$$\Rightarrow \quad \theta = \cos^{-1}\frac{1}{3}.$$

28. Let A and B be two given points (3, 2, –1) and (–4, 6, 3) respectively.

Let C be the point (1, y, 2)

The d.rs of AB are $\langle -4-3, y-2, 3-(-1)\rangle$

$= \quad \langle -7, 4, 4\rangle$

The d.rs of AC are $\langle 1-3, y-2, z+1\rangle$

$= \quad \langle -2, y-2, z+1\rangle$

Since A, C & B are collinear, we have

$$\frac{-2}{-7} = \frac{y-2}{4} = \frac{z+1}{4}$$

$$\Rightarrow \quad \frac{y-2}{4} = \frac{z+1}{4} = \frac{2}{7}$$

$$\Rightarrow \quad y = \frac{22}{7}, z = \frac{1}{7}$$

29. Three given planes are

$x + 2y + 3z - 4 = 0$ ....(1)

$2x + y - z + 5 = 0$ ....(2)

$2x - y + 2z - 3 = 0$ .....(3)

Equation of the plane passing through the intersection of (1) and (2) is

$x + 2y + 3z - 4 + k(2x + y - z + 5) = 0$

$\Rightarrow$ $(1+2k)x + (2+k)y + (3-k)z + (5k-4) = 0$ ....(4)

Since this plane is perpendicular to the plane (3), we have

$(1+2k)2 + (2+k)(-1) + (3-k)2 = 0$

$\Rightarrow$ $k = -6$

Required plane is

$x + 2y + 3z - 4 - 6(2x + y - z + 5) = 0$

$\Rightarrow$ $11x + 4y - 9z + 3y = 0.$

30. Let P be the point (a, b, c).

From P, let us draw perpendiculars PA, PB and PC to xy-plane, yz-plane and zx-plane.

The coordinates of A, B and C are (a, b, o), (o, b, c) and (a, o, c).

Equation of the plane passing through A, B and C is

$$\begin{vmatrix} x-a & y-b & z-o \\ o-a & b-b & c-o \\ a-a & o-b & c-o \end{vmatrix} = o$$

$$\Rightarrow \begin{vmatrix} x-a & b-b & c^z \\ -a & o & c \\ o & -b & c \end{vmatrix} = o$$

$\Rightarrow$ $bcx + acy + ab\,z = 2abc$

$$\Rightarrow \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 2$$

31. The given points are (2, 1, -1) and (-1, 3, 4).

Given plane is $x-2y+4z=10$ ....(1)

Equation of the plane passing through (2, 1, -1) is $a(x-2) + b(y-1) + c(z+1) = 0$ .....(2)

Since this plane is passing through (-1, 3, 4), we have $a(-1-2) + b(3-1) + c(4+1) = 0$

$\Rightarrow$ $-3a + 2b + 5c = 0$ ....(3)

Since the plane (2) is perpendicular to (1), we $a.1 + b(-2) + c.4 = 0$

$\Rightarrow$ $a - 2b + 4c = 0$ ....(4)

From (3) e (y), by cross multiplication.

$$\frac{a}{8+10} = \frac{b}{5+12} = \frac{c}{6-2}$$

$$\Rightarrow \quad \frac{a}{18} = \frac{b}{17} = \frac{c}{4} = k \quad \text{(say)}$$

$\Rightarrow$ $a = 18K, \; b = 17K, \; c = 4K.$

Required plane is

$18K(x-2) + 17K(y-1) + 4K(z+1) = 0$

$\Rightarrow$ $18(x-2) + 17(y-1) + 4(z+1) = 0$

$=$ $18x + 17y + 4z - 49 = 0$

32. The given line is

$$\frac{x-1}{1}=\frac{y+2}{3}=\frac{z-1}{-1}=r \qquad ......(1)$$

The given plane is

$$2x + y + z = 9 \qquad .....(2)$$

Let the line (1) intersect the plane (2) at P.

Any point on the line (1) is

$$(r + 1,\ 3r–2,\ –r+1)$$

The coordinates of P are also (r+1,3r –2,–r+1) for some value of r.

Then this will satisfy the plane (2),

$$\therefore \quad 2(r+1)+3r-2+(-r+1)=9$$

$$\Rightarrow 4r = 8 \ \Rightarrow \ r = 2.$$

33. Let A and B be two points whose coordinates are (1, 3, -1) and (2, 6, –2) respectively.

Let zx-plane divides AB is the ratio k : 1 at C.

The coordinates of C are

$$= \left(\frac{K.2+1}{K+1},\frac{K.6+3}{K+1},\frac{K(-2)-1}{K+1}\right)$$

$$= \left(\frac{2K+1}{K+1},\frac{6K+3}{K+1},\frac{-2K-1}{k+1}\right)$$

Since it is a point zx–plane,

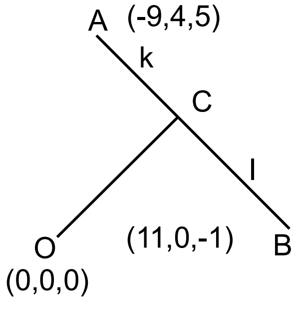
we have $\frac{6K+3}{K+1}=0$

$$\Rightarrow 6K + 3 = 0 \qquad \Rightarrow 6K = -3$$

$$\Rightarrow K = –3 = -\frac{1}{2}.$$

zx=plane divides AB in the ratio –1 : 2.

34. Let A and B be two points whose coordinates are (-9, 4, 5) and (11,0,-1) respectively.



Let OC be the perpendicular from the origin O to AB.

Let C divides AB in the ratio K : 1.

The coordinates of C are

$$\left(\frac{11K-9}{K+1},\frac{4}{K+1},\frac{-K+5}{k+1}\right)$$

The d.rs of OC are $\left\langle\frac{11K-9}{k+1},\frac{4}{K+1},\frac{-K+5}{K+1}\right\rangle$

The d.rs of OC are

$$\left\langle\frac{11K-9}{k+1}-0,\frac{4}{K+1}-0,\frac{-K+5}{K+1}-0\right\rangle$$

$$= \left\langle\frac{11K-9}{k+1},\frac{4}{K+1},\frac{-K+5}{K+1}\right\rangle$$

The d.rs of AB are $\langle 11-(-9),0-4,-1-5\rangle$

$$= \langle 20,-4,-6\rangle$$

Since OC is perpendicular to AB, we have

$$20\left(\frac{11K-9}{K+1},\frac{4}{K+1},\frac{-K+5}{K+1}\right) = 0$$

$$\Rightarrow 20\ (11K –9) – 16 + 6K – 30 = 0$$

$$\Rightarrow K = 1$$

The coordinates of C are

$$\left(\frac{11.1-9}{1+1},\frac{4}{1+1},\frac{-1+5}{1+1}\right)$$

$$= (1, 2, 2)$$

35. Let P be the given point (2, -1, 3)

The given plane is

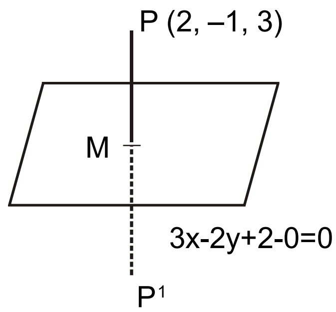
$$3x – 2y + 2 – 9 = 0 \qquad ....(1)$$

From, P let us draw a perpendicular PM to the plane (1).

$\therefore$ P' is the image of P.

The d.rs of PM are $\langle 3,-2,1\rangle$.

Equation of the live PMP' is



$$\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-3}{1} = r \text{ (say)}$$

Any point on thus line is (3r+2, –2r-1, r+3)

The coordinates of P' are (3r + 2, –2r-1, r+3) for some value of r.

Since M is the middle point of PP', its coordinates are

$$\left(\frac{3r+2+2}{2},\frac{-2r-1-1}{2},\frac{r+3+3}{2}\right)$$

$$= \quad \frac{3r+4}{2},\frac{-2r-2}{2},\frac{r+6}{2}$$

Since it is a point on the plane (1)m we have

$$3\left(\frac{3r+4}{2}\right)-2\left(-\frac{2r-2}{2}\right)+\frac{r+6}{2}-9=0$$

$\Rightarrow$ 3 (3r + 4) + (2r+2)+r+6 – 18 = 0

$\Rightarrow$ 9r + 12 + 2r + 2 + r – 12 = 0

$\Rightarrow$ 12r = –2

$$\Rightarrow \quad r=-\frac{2}{12}=-\frac{1}{6}$$

The coordinates of P' are

$$\left(3\left(-\frac{1}{6}\right)+2,\ -2\left(\frac{1}{6}\right)-1,\frac{-1}{6}+3\right)$$

$$= \quad \left(-\frac{1}{2}+2,\ \frac{1}{3}-1,\ \frac{-1+18}{6}\right)$$

$$= \quad \left(\frac{3}{2},\frac{-2}{3},\frac{7}{6}\right)$$

36. Given line is $\frac{x-2}{1}=\frac{y}{-1}=\frac{z-1}{2}=r$ (say) ...(1)

Given plane is 2x+y+z=2 ....(2)

Let the line (1) intersect the plane (2) at M.

Any point on the line (1) is

(r+2, –r, 2r + 1)

The coordinates of M are also (r+2, -r, 2r+1) for some value of r.

$\therefore$ $2(r+2)-r+2r+1=2$

$\Rightarrow$ 2r + Y - r + 2r + 1 = 2

$\Rightarrow$ 3r = –3 $\Rightarrow$ r = -1

The coordinates of M are (-1+2, –(–1), 2(-1)+1)
= (1, 1, -1).

37. The equation of the plane is

3x + 3z - 5 = 0

$\Rightarrow$ 3x + OY + 3z - 5 = 0 ....(1)

The d.rs of the normal to the plane (1) are

$\langle 3, 0, 3\rangle$.

Equation of the line is

$$\frac{x-1}{1}=\frac{y-2}{-1}=\frac{z-3}{0} \quad ....(2)$$

The d.rs of the line are $\langle 1,-1,0\rangle$.

Let θ be the angle between the plane (1) and the line (2).

The angle between the line (2) and the normal of the plane (1) = 90 - θ.

$\therefore$ $\cos\theta(90-\theta)$

$$= \frac{3.1+0.(-1)+3.0}{\sqrt{3^2+0^2+3^2}\sqrt{1^2+(-1)^2+0^2}}$$

$$\Rightarrow \quad \sin\theta=\frac{3}{\sqrt{18}\sqrt{2}}=\frac{3}{3\sqrt{2}\sqrt{2}}=\frac{1}{2}$$

$$\Rightarrow \quad \theta=\frac{\pi}{6}.$$

The angle between the plane and the line (2) is $\frac{\pi}{6}$.

38. The given line is

x + 2y - z = 5 ......(1)

Given point is (1, 1, 2).

The (1) is x + 2y - z = C .....(2)

Since it is passing through (1, 1, 2), we have

1 + 2 - 2 = C $\Rightarrow$ C = 1.

Required plane is x + 2y – z = 1.

39. Let A and B be two given points whose coordinates are (-1, 4, 3) and (5, -2, -1) respectively.

Let C be the middle point of AB.

The coordinates of C are $\left(\frac{-1+5}{2},\frac{4-2}{2},\frac{3-1}{2}\right)$

$=$ (2, 1, 1).

The d.rs of AB are $\langle 5-(1),-2-4,-1-3\rangle$

$= \langle 6,-6,-4\rangle$

The equation of the plane is

$6(x-2)+(-6)(y-1)+(-4)(z-1)=0$

$\Rightarrow 6(x-2)-6(y-1)-(4)(z-1)=0$

$\Rightarrow 3(x-2)-3(y-1)-2(z-1)=0$

$\Rightarrow 3x-3y-2z-1=0$

40. Two given planes are

$3x-2y+6z-7=0$ .....(1)

$3x-2y+6z-14=0$ .....(2)

Let us take a point on the line (1) where it intersect x-axis i.e., when y = 0, z = 0.

From (1), we get $3x-7=0$

$\Rightarrow x=\frac{7}{3}$.

A point on the plane (1) is $\left(\frac{7}{3},0,0\right)$.

The distance between two planes

= The distance from $\left(\frac{7}{3},0,0\right)$ to the plane (2)

$$=\frac{\left|3.\frac{7}{3}-2.0+6.0-14\right|}{\sqrt{3^2+(-2)^2+6^2}}$$

$$=\frac{|-7|}{\sqrt{9+4+36}}=\frac{7}{7}=1$$

41. The given plane is

$2x+y+z=9$ .....(1)

Given line is $3x-3=y+2=3-3z$

$\Rightarrow 3(x-1)=y+2=-3(z-1)$

$\Rightarrow \frac{x-1}{1}=\frac{y+2}{3}=\frac{z-1}{-1}=r$ (say) ....(2)

Any point on the line (2) is (r+1, 3r-2, –r+1).

The coordinates of the point of intersection of the line (2) & the plane (1) is also

(r+1, 3r–2, –r+1) for some value of r.

$\therefore 2(r+1)+3r-2-r+1=9$

$\Rightarrow r=2$

Required point is (2+1, 3.2-2, –2+1)

= (3, 4, -1).

42. The given points are (–2, 3, 5), (7, –7, -5) and (–2, 5, –3).

Equation of the plane passing through above three points is

$$\begin{vmatrix} x-(-2) & y-3 & z-5 \\ 7-(2) & -7-3 & -5-5 \\ -2-(-2) & 5-3 & -3-5 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} x+2 & y-3 & z-5 \\ 9 & -10 & -10 \\ 0 & 2 & -8 \end{vmatrix}=0$$

$\Rightarrow (x+2)(80+20)-(y-3)[-72-0]+(z-5)(18-0)=0$

$\Rightarrow 100(x+2)+72(y-3)+18(z-5)=0$

$\Rightarrow 50(x+2)+36(y-3)+9(z-5)=0$

43. Two given planes are

$3x+y-z-2=0$ .....(1)

$x-y+2z-1=0$ ....(2)

The given point is (1, 0, 2).

Equation of the plane passing through the intersection of (1) and (2) is

$3x + y - z - 2 + K(x - y + 2z - 1) = 0$

Since it is passing through (1, 0, 2), we have

$3 + 0 - 2 - 2 + K(1 - 0 + 4 - 1) = 0$

$\Rightarrow -1 + K.4 = 0$

$\Rightarrow K = \frac{1}{4}$.

Required plane is

$3x + y - z - 2 + \frac{1}{4}(x - y + 2z - 1) = 0$

$\Rightarrow 12x + 4y - 4z - 8 + x - y + 2z - 1 = 0$

$\Rightarrow 13x + 3y - 2z - 9 = 0$

44. Given plane is

$x - 2y + 4z = 11$ .....(1)

Two given points are (1, 3, –2) and (3, 4, 1).

Equation of the line passing through (1, 3, -2) and (3, 4, 1) is

$$\frac{x-1}{3-1} = \frac{y-3}{4-3} = \frac{z-(-2)}{1-(-2)}$$

$$\Rightarrow \frac{x-1}{2} = \frac{y-3}{1} = \frac{z+2}{3} = r \text{ (say)} \quad ....(2)$$

Let the point of intersection of the line (2) and the plane (1) be P.

Any point on the line (2) is

(2r+1, r+3, 3r-2)

The coordinates of P are also (2r+1, r+3, 3r-2) for the same value of r.

Then this point will satisfy the plane (1)

$\therefore 2r + 1 - 2(r + 3) + 4(3r - 2) = 11$

$\Rightarrow 2r + 1 - 2r - 6 + 12r - 8 = 11$

$\Rightarrow 12r = 24$

$\Rightarrow r = 2$

The coordinates of P are (2.2+1, 2+3, 3.2-2)

= (5, 5, 4)

45. The unsymmetric form of the line is

$$\left.\begin{aligned} x = 2y + z - 3 = 0 \\ 6x + 8y + 3z - 1 = 0 \end{aligned}\right\} .... (1)$$

Let us take that point on the line where it intersect xy-plane. i.e., where z = 0.

From (1), we get $x + 2y - 3 = 0$

$6x + 8y - 1 = 0$

solving the above two equations, we get

$$\frac{x}{-2+24} = \frac{y}{-18+1} = \frac{1}{8-12}$$

$$\Rightarrow \frac{x}{z^2} = \frac{y}{-17} = \frac{1}{-4}$$

$$\Rightarrow x = -\frac{22}{4},\ y = \frac{17}{4}$$

A point on the line is $\left(-\frac{22}{4}, \frac{17}{4}, 0\right)$

Let the d.r s of the line (1) be < a, b, c >

The d.r.s of the normals of two planes in (1) are < 1, 2, 1 > and <6, 8, 3).

$\therefore a + 2b + c = 0$

$6a + 8b + 3C = 0$

By cross multiplication, we get

$$\frac{a}{6-8} = \frac{b}{6-3} = \frac{c}{8-12}$$

$$\Rightarrow \frac{a}{-2} = \frac{b}{3} = \frac{c}{-4}$$

The line is passing through $\left(-\frac{22}{4}, \frac{17}{4}, 0\right)$ and whose d.r s are < –2, 3, –4 >.

Thus the equation of the line is

$$\frac{x - \left(-\frac{22}{44}\right)}{-2} = \frac{y - \frac{17}{4}}{3} = \frac{z-0}{-4}$$

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GROUP - C

LONG QUESTIONS

Each questions carries 6 marks

7. Relations and Functions, Inverse Trigonometric functions, Linear Programming.

1. If $f: x \to y$ and $g: x \to z$ are two fucntions, show that gof is invertible if each of f and g is so and then $(gof)^{-1} = f^{-1}og^{-1}$.

2. Prove that $f: x \to y$ is enjective iff for all subsets A, B of X, $f(A \cap B) = f(A) \cap f(B)$.

3. If $f: A \to B$ and $g: B \to A$ such that gof is an identity function on A and fog is an identity function on B, then show that g = f^{-1}.

4. If p is a prime and $ab \equiv 0$ (mod p) then show that either $a \equiv 0$ (mod p) or $b \equiv 0$ (mod p).

5. It A = {1, 2, 3, ...9} and R be the relation in $A \times A$ defined by

   (a, b) R (c, d) as a + d = b + c for (a,b), (c,d) $\in A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class [(2,5)].

6. If $f: R \to R$ defined by f(x) = 5x–8 for all $x \in R$, then show that f is convertible. Find the corresponding inverse function.

7. If $f(x) = \cos[\pi^2]x + \cos[-\pi^2]x$, where [x] stands for the greatest integer functions, then evaluate $+\left(\frac{\pi}{2}\right), f(\pi)$, $f(-\pi)$ and $f\left(\frac{\pi}{4}\right)$.

8. If ABC is a right-angled triangle at A prove that $\tan^{-1}\left(\frac{b}{a+c}\right) + \tan^{-1}\left(\frac{c}{a+b}\right) = \frac{\pi}{4}$ where a, b, c are the sides of a triangle.

9. If $\cos^{-1}\frac{x}{a} + \cos^{-1}\frac{y}{b} = \theta$ then prove that $\frac{x^2}{a^2} - \frac{2xy}{ab}\cos\theta + \frac{y^2}{b^2} = \sin^2\theta$.

10. Prove that $\cos^{-1}\left(\frac{a\cos x + b}{a + b\cos x}\right)$

    $= 2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)$

11. If $\sin^{-1}\left(\frac{x}{a}\right) + \sin^{-1}\left(\frac{y}{b}\right) = \sin^{-1}\left(\frac{c^2}{ab}\right)$ then prove that $b^2x^2 + 2xy\sqrt{a^2b^2 - c^4} + a^2y^2 = c^4$.

12. Prove that $\tan^{-1}\left[\frac{\sqrt{1+x} - \sqrt{1-x}}{\sqrt{1+x} + \sqrt{1-x}}\right]$

    $= \frac{\pi}{4} - \frac{1}{2}\cos^{-1}x \left(-\frac{1}{\sqrt{2}} \le x \le 1\right)$

13. Solve the following L.P.P. graphically

    Maximize $Z = 4x_1 + 3x_2$

    Subject to $x_1 + x_2 \le 50$

    $x_1 + 2x_2 \le 80$

    $2x_1 + x_2 \ge 20$

14. Solve the following L.P.P.

    Maximize $Z = 20x + 10y$

    Subject to $x + 2y \le 40$

    $3x + y \ge 30$

    $4x + 3y \ge 60$

    $x, y \ge 0$

15. Solve the following L.P.P.

    Minimize $Z = 20x_1 + 40x_2$

    Subject to $36x_1 + 6x_2 \ge 108$

    $3x_1 + 12x_2 \ge 36$

    $2x_1 + x_2 \ge 10$

    $x_1, x_2 \ge 0$

8. Matrices, Determinants, Probability. Each question carries 6 marks.

1. Exmine the consistency and solvability, solve the following equations by matrix method.

$$x - 2y = 3$$
$$3x + 4y - z = -2$$
$$5x - 3z = -1$$

2. Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \\ 1 & 0 & 2 \end{bmatrix}$ by using elementary transformation.

3. Verify that $(AB)^T = B^T A^T$

where $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 6 & -3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{bmatrix}$

4. If $A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{bmatrix}$ then find A^{-1}.

Using A^{-1}, solve the following system of equations.

3x–2y+2=2

2x+y–3z=–5

–x+2y+z=6

5. Determine the matrices A and B where

$$A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix} \text{ and } 2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

6. Prove that $\begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

$= (a^2 + b^2 + c^2)(a+b+c)(b-c)(c-a)(a-b)$

7. Prove that

$$\begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix} = (1+a^2+b^2)^3$$

8. Prove that $\begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

$= a^3 + b^3 + c^3 - 3abc$

9. If $x \neq y \neq z$ and

$$\begin{vmatrix} x & x^2 & 1+x^3 \\ y & y^2 & 1+y^3 \\ z & z^2 & 1+z^3 \end{vmatrix} = 0 \text{ then } 1+xyz=0$$

10. Using the properties of the determinant, Prove that

$$\begin{vmatrix} 2y & y-z-x & xy \\ 2z & 2z & z-x-y \\ x-y-z & 2x & 2x \end{vmatrix} = (x+y+z)^3$$

11. There are 3 bags B_1, B_2 and B_3 having respectively 4 white, 5 black; 3 white 5 black and 5 white. 2 black balls. A bag is chosen at random and a ball is drawn from it. Find the probability that the ball is white.

12. Find the probability distribution of number of doublets in four throws of a pair of dice. Find also the mean and varcance of the number of doublet.

13. A pair of dice is thrown 5 times then find the probability of getting three doublets.

14. A box containing 20 electric bulbs includes 5 detective bulbs. Four bulbs are drawn at random with replacement. Find the probability distribution of the number of non - defective bulbs. Calculate also the mean and the varuance.

15. Five boys and four girls randomly stand in a line. Find the probability that no two girls come together.

9. Continuity and differentiability, Applications of derivatives. Each question carries 6 marks.

1. If $e^{\frac{y}{x}} = \frac{x}{a+bx}$ then show that

$$x^3 \frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(x\frac{dy}{dx} - y\right)^2$$

2. Find the value of K for which

$$f(x) = \begin{cases} \dfrac{\sqrt{1+kx} - \sqrt{1-kx}}{x} & \text{when } 1 < x < 0 \\ \dfrac{2x+1}{x-1} & \text{when } 0 \le x < 1 \end{cases}$$

is continuous at x = 0.

3. If $x = \frac{1-\cos^2\theta}{\cos\theta}$, $y = \frac{1-\cos^{2n}\theta}{\cos^n\theta}$, then show that $\left(\frac{dy}{dx}\right)^2 = n^2\left(\frac{y^2+4}{x^2+4}\right)$.

4. Find $\frac{dy}{dx}$ when $y = \cot^{-1}\left(\ln\cos e^{-x}\right) + \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$

5. Find $\frac{dy}{dx}$ if $x^y = y^x + \tan^{-1}\frac{\cos x}{1+\sin x}$

6. Differentiate $\tan^{-1}\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}$ with respect to $\ln\left(\frac{1+\cos x}{1-\cos x}\right)$.

7. If $y = \frac{\sin^{-1}x}{\sqrt{1-x^2}}$ then show that

$$\left(1-x^2\right)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$

8. If $y = x^{\sin x - \cos x} + \frac{x^2-1}{x^2+1}$ then find $\frac{dy}{dx}$.

9. Show that the curves y = 2^x nd y = 5^x intersect at an angle $\tan^{-1}\left[\frac{\ln 5/2}{1+\ln 2\ln 5}\right]$

10. Find the maximum value of $f(x) = x^{\frac{1}{x}}, x > 0$ and show that $e^{\pi} > \pi^{e}$

11. Show that teh sum of x-intercept and y-intercept of any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{a}$ is constant.

12. Find the altitude of the right circular cylinder of maximum volume that can be inscribed within a sphere of radius R.

13. Show that the semi-vertical angle of the cone of maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$.

14. Find the point on the curve $y^2 - x^2 + 2x - 1 = 0$ where the tangent is parallel to the curve.

15. Show that the minimum distance of a point on the curve $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ from the origin is a+b.

10. Integrals, Application of Integrals, Differential Equations. Each question carries 6 marks.

1. Integrate $\int \frac{dx}{\cos x(1+2\sin x)}$

2. Evaluate $\int_0^{\pi} \frac{x}{1+\sin x}dx$

3. Show that $\int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt[3]{\cot x}}dx = \frac{\pi}{6}$

4. Show that $\int_0^1 \frac{\log(1+x)}{1+x^2}dx = \frac{\pi}{8}\log 2$

5. Evaluate $\int \frac{dx}{2\cos^2 x + 3\cos x}$

6. Evaluate $\int_0^{\pi/2} \frac{\sin x(7-\cos x)}{\left(1+\cos^2 x\right)\left(2-\cos^2 x\right)}dx$

7. Determine the area included between the parabola $y^2 = x$ and the circle $x^2 + y^2 = 2x$.

8. Find the area enclosed by y=4 x–1 and $y^2 = 2x$.

9. Find the area of the portion of the ellipse $\frac{x^2}{12} + \frac{y^2}{16} = 1$ bounded by the major axis and the double ordinate x = 3.

10. Find the area enclosed by the parabola $y^2=4x$ and the line y = 2x.

11. Solve $\frac{dx}{dy} = \frac{3x-7y+7}{3y-7x-3}$.

12. Solve the differential equation $x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\frac{y}{x}=0$. Find the particular solution of this differential equation given that x = 1 shen $y=\frac{\pi}{2}$

13. Solve $xdy-ydx=\sqrt{x^2+y^2}dx$

14. Solve $\frac{dy}{dx}-y\cot x=xy^4$

15. $\frac{dy}{dx}+2y\tan x=\sin x,\ y\left(\frac{\pi}{3}\right)=0$

11. Vectors, Three dimensional Geometry. Each question carries 6 marks.

1. If $\vec{a}=2\hat{i}+\hat{k},\vec{b}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{c}=4\hat{i}-3\hat{j}+7\hat{k}$ then find vector $\vec{r}$ which satisfies $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}$ and $\vec{r}.\vec{a}=0$.

2. Prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right)+\vec{b}\times\left(\vec{c}\times\vec{a}\right)+\vec{c}\times\left(\vec{a}\times\vec{b}\right)=\vec{0}$ and hence prove that $\vec{a}\times\left(\vec{b}\times\vec{c}\right),\vec{b}\times\left(\vec{c}\times\vec{a}\right)$ and $\vec{c}\times\left(\vec{a}\times\vec{b}\right)$ are coplanar.

3. If $\vec{a},\vec{b}$ and $\vec{c}$ are three vectors such that $\left|\vec{a}\right|=5$, $\left|\vec{b}\right|=12$ and $\left|\vec{c}\right|=13$ and $\vec{a}+\vec{b}+\vec{c}=0$ then find the value of $\vec{a}.\vec{b}+\vec{b}.\vec{c}+\vec{c}.\vec{a}$.

4. Prove by vector method that is any triangle ABC, a = b cos c + c cos B

5. Prove by vector method that is any triangle ABC, $a^2=b^2+c^2-2bc\cos A$.

6. By vector method, find the area of the triangle whose vertices are A(2, -3, 5), B (3, 0, 7) and c (4, 0, 6). Also find m $\angle ABC$.

7. Obtain the volume of the parallelopiped whose sides are vectors $\vec{a}=2\hat{i}-3\hat{j}+4\hat{k}$, $\vec{b}=\hat{i}+2\hat{j}-\hat{k}$ and $\vec{c}=3\hat{i}-\hat{j}+2\hat{k}$. Also find the vector $\left(\vec{a}\times\vec{b}\right)\times\vec{c}$.

8. Find the vector $\vec{p}$ which is perpendicular to both $\vec{a}=4\hat{i}+5\hat{j}-\hat{k}$ and $\vec{\beta}=\hat{i}-4\hat{j}+5\hat{k}$ and $\vec{p}.\vec{q}=21$ where $\vec{q}=3\hat{i}+\hat{j}-\hat{k}$.

9. Prove that the lines $\frac{x+3}{2}=\frac{y+5}{3}=\frac{z-7}{-3}$ and $\frac{x+1}{4}=\frac{y+1}{5}=\frac{z+1}{-1}$ are coplanar. Find the equation of the plane containing them.

10. Prove that the four points (0, 4, 3), (-1, -5, -3), (-2, -2, 1) and (1, 1, –1) lie in one plane. Find the equation of the plane.

11. A variable plane is at a constant distance 3r from the origin and meets the axes at A, B and C. Show that the locus of the centroid of the triangle ABC is $x^{-2}+b^{-2}+2^{-2}=r^{-2}$.

12. If the edges of a rectangular parallelopiped are of lengths a, b, c, then the angle between four diagonals are $\cos^{-1}\left(\frac{\pm a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right)$.

13. If l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of two mutually perpendicular lines, show that the d.cs of the line perpendicular to both of them are m_1n_2 - m_2n_1, $n_1l_2–n_2l_1$, l_1m_2 - l_2m_1.

14. Find the equation of the plane through the points (2, 2, 1) and (9, 3, 6) and perpendicular to the plane 2x+6y+6z=0.

15. Prove that the line joining (1, 2, 3) and (2, 1, -1) intersect the line joining (–1, 3, 1) and (3, 1, 5).

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GROUP - C

ANSWERS

7. Relations and Functions, Inverse Trigonometric functions, Linear Programming.

1. Two given functions are

$f: X \to Y$ and $g: Y \to Z$

First we shall show that if f and g are invertible then gof is also invertible.

i.e., if f and g are one-one and on to then gof is also one-one and on to.

(i) Let f and g are one-one function.

$\therefore$ For $x_1, x_2 \in x$,

(gof) (x_1) = (gof) (x_2)

$\Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow f(x_1) = f(x_2)$ ($\therefore$ g is one-one)

$\Rightarrow x_1 = x_2$ ($\because$ f is one-one)

Thus for $x_1, x_2 \in x$,

(gof) (x_1) = (gof) (x_2) $\Rightarrow x_1 = x_2$

$\Rightarrow$ gof is one-one

(ii) Let f and g be two on to functions.

Let $z \in Z$.

Given that $g: Y \to Z$ is an on to function.

So there exists an element $y \in Y$ such that

g(y) = z

Since $f: X \to Y$ be on to, then there exists an element $x \in x$ such that $f(x) = y$

Now $g(y) = z$

$\Rightarrow g[f(x)] = z$

$\Rightarrow (gof)(x) = z$

Thus for any element $z \in Z$, there exists $x \in X$ such that (gof)(x) = z.

$\Rightarrow$ $(gof): X \to Z$ is an on to function.

Thus we see that if f and g are one-one and on to then gof is also an one-one and on to function

$\Rightarrow$ gof is invertible

$\Rightarrow$ $(gof)^{-1}$ exists.

Here $f: X \to Y$ is bijective.

There exists $y \in Y$ such that $f(x) = y$

$\Rightarrow x = f^{-1}(y)$ ......(1)

Again given that $g: Y \to Z$ is bijective.

So there exists an element $z \in Z$ such that

g(y) = z

$\Rightarrow$ y = $g^{-1}(z)$ .....(2)

Now $(gof)(x) = g[f(x)] = g(y) = z$

$\Rightarrow x(gof)^{-1}(z)$ ....(3)

Also $x = f^{-1}(y) = f^{-1}[g^{-1}(2)]$

$= (f^{-1}og^{-1})(z)$ ......(4)

From (3) & (4), we see that

$(gof)^{-1} = f^{-1}og^{-1}$.

2. Given that $f: X \to Y$ is injective.

Let A and B are subsets of X.

Let $f(x) \in f(A \cap B)$

$\Leftrightarrow x \in A \cap B$

$\Leftrightarrow x \in A$ and $x \in B$

$\Leftrightarrow f(x) \in f(A)$ and $f(x) \in f(B)$

$\Leftrightarrow f(x) \in f(A) \cap f(B)$

Conversely suppose that

$f(A \cap B) = f(A) \cap (B)$

Let f is not injective

If $f(x) \in f(A \cap B) \Leftrightarrow x \in A \cap B$

$\Leftrightarrow$ $x \in A$ and $x \in B$

$\not\Leftrightarrow$ $f(x) \in f(A)$ and $f(x) \in f(B)$

$\Leftrightarrow$ $f(x) \in f(A) \cap f(B)$

$\therefore$ $f(A \cap B) = f(A) \cap f(B)$ is false.

So f must be injective.

3. Given that gof = I_A and fog = I_B
we have to show that $g = f^{-1}$.

First we shall show that f is invertible

i.e., we shall show that f is one-one and onto.

For $x_1, x_2 \in A$, $f(x_1) = f(x_2)$

$\Rightarrow g[f(x_1)] = g[f(x_2)]$

$\Rightarrow (gof)(x_1) = (gof)(x_2)$

$\Rightarrow I_A(x_1) = I_B(x_2)$

$\Rightarrow x_1 = x_2$

This show that f is one-one.

Again Let $y \in B$ and let $g(y) = x$

$\therefore$ $g(y) = x$

$\Rightarrow$ $f[g(y)] = f(x)$

$\Rightarrow$ $(fog)(y) = f(x)$

$\Rightarrow$ $I_B(y) = f(x)$

$\Rightarrow$ $y = f(x)$

For each $y \in B$, there exists an $x \in A$ such that $f(x) = y$

So f is on to.

$\Rightarrow$ f is invertible.

Now jog = I_B

$\Rightarrow$ $f^{-1}o(fog) = (f^{-1}of)B = f^{-1}$

$\Rightarrow$ $(f^{-1}of)og = f^{-1} = I_A og = f^{-1}$

$\Rightarrow$ $g = f^{-1}$

4. Given that p is prime

Also given that $ab \equiv o(\text{mod } p)$

$\Rightarrow$ p divides ab.

$\Rightarrow$ There exists $c \in Z$ such that

ab = pc .....(1)

we shall show that either $a \equiv 0$ $(\text{mod } p)$

or $b \equiv 0$ (mod p)

If possible let $a \equiv 0$ (mod p)

$\Rightarrow$ (p, a) = 1

($\because$ p is prime and a is relatively prime g.c.f of p and a is 1)

$\Rightarrow$ There exists $m_1 n \in Z$ such that pm+an=1

Multiplying both sides by b, we get

pmb + anp = b

$\Rightarrow$ $b\,mp + an\,p = b$

$\Rightarrow$ b mp + p cn = b.

$\Rightarrow$ p(bm+cn) = b

p divides b

$\Rightarrow$ $b \equiv 0(\text{mod } p)$

Similarity if $b \equiv 0$ (mod p) then we can show that $a \equiv 0$ (mod p)

5. Given that A={1,2,3,.........9}

The relation R in $A \times A$ is defined for (a,b), (c,d) $\in A \times A$,

$(a,b)R(c,d) = a + d = b + c$.

We observe the following proporties on R.

Reflexive

Let (1, 2) be an element of $A \times A$.

(1, 2) $\in$ $A \times A$ $\Rightarrow$ 1, 2 $\in$ A

1+2 = 2 +1 (Addition is commutative)

$\Rightarrow$ (1, 2) R (1, 2)

Thus for all $(1,2) \in A \times A$, $(1,2)R(1,2)$.

So the relation R is reflexive $mA \times A$.

Symmetric

Let (1, 2), (3, 4) $\in A \times A$ such that

(1, 2) R (3, 4)

$\Rightarrow$ 1 + 4 = 2 + 3

$\Rightarrow$ 3 + 2 = 4 + 1

$\Rightarrow$ 3 + 2 = 1 + 4

$\Rightarrow$ (3, 4) R (1, 2)

Thus, (1, 2) R (3, 4) $\Rightarrow$ (3, 4) R (1, 2).

$\Rightarrow$ R is symmetric.

Transitive

Let (1, 2), (3, 4), (5, 6) $\in A \times A$.

such that (1, 2) R (3, 4), (3, 4) R (3, 6).

Now (1, 2) R (3, 4) $\Rightarrow$ 1 + 4 = 2 + 3

Again (3, 4) R (5, 6) $\Rightarrow$ 3 + 6 = 4 + 5

$\therefore$ (1+4) + (3+6) = (2+3) + (4+5)

$\Rightarrow$ 1 + 6 = 2 + 5

$\Rightarrow$ 1 + 6 = 5 + 2

$\Rightarrow$ (1, 2) R (5, 6)

So R is transitive.

Therefore R is an equivalence relation.

The equivalence class of (2, 5)

[(2, 5)] = {(1, 4), (2, 5), (3, 6), (4,7), (5,6),(6, 9)}

6. The given function is $f : R \to R$ and is defined by f(x) = 5x – 8.
Let $x_1, x_2 \in$ Domain R.

$\therefore$ $f(x_1) = f(x_2)$

$\Rightarrow$ $5x_1 - 8 = 5x_2 - 8$

$\Rightarrow$ $5x_1 = 5x_2$

$\Rightarrow$ $x_1 = x_2$

So f is one-one.

Let $y \in$ Range of R such that

y = 5x – 8

$\Rightarrow$ 5x = y + 8

$\Rightarrow$ $x = \frac{y+8}{5}$

Thus any $y \in$ Range R is the image of $\frac{y+8}{5} \in$ domain R

So f is on to

Since f is one-one and on to, f is bijective.

$\Rightarrow$ f is invertible.

$\therefore$ $f^{-1}(y) = \frac{y+8}{5}$

$\Rightarrow$ $f^{-1}(x) = \frac{x+8}{5}$

7. We know $\left[\pi^2\right] = 9$

$\left[-\pi^2\right] = -10$

$f(x) = \cos\left[\pi^2\right]x + \cos\left[-\pi^2\right]x$

$= \cos 9x + \cos(-10x)$

$= \cos 9x + \cos 10x$

$\therefore$ $f\left(\frac{\pi}{2}\right) = \cos\frac{9\pi}{2} + \cos\frac{10\pi}{2}$

$= \cos\left(4\pi + \frac{\pi}{2}\right) + \cos 5\pi$

$= \cos\frac{\pi}{2} + \cos\pi = 0 - 1 = -1$

$f(\pi) = \cos 9\pi + \cos 10\pi$

$= \cos(8\pi + \pi) + 1$

$= \cos\pi + 1 = -1 + 1 = 0$

$f(-\pi) = \cos(-9\pi) + \cos(-10\pi)$

$= \cos 9\pi + \cos 10\pi = 0$

$f\left(\frac{\pi}{4}\right) = \cos\frac{9\pi}{4} + \cos\frac{10\pi}{4}$

$= \cos\left(2\pi + \frac{\pi}{4}\right) + \cos\left(2\pi + \frac{\pi}{2}\right)$

$= \cos\frac{\pi}{4} + \cos\frac{\pi}{2}$

$= \frac{1}{\sqrt{2}} + 0 = \frac{1}{\sqrt{2}}$

8. $\angle A = 90^0$

$\therefore \quad a^2 = b^2 + c^2$

L.H.S $= \tan^{-1}\dfrac{b}{a+c} + \tan^{-1}\dfrac{c}{a+b}$

$= \tan^{-1}\left(\dfrac{\dfrac{b}{a+c}+\dfrac{c}{a+b}}{1-\dfrac{b}{a+c}.\dfrac{c}{a+b}}\right)$

$= \tan^{-1}\left(\dfrac{ab+ac+b^2+c^2}{a^2+ac+ab}\right)$

$= \tan^{-1}\left(\dfrac{ab+ac+a^2}{a^2+ac+ab}\right) = \tan^{-2} 1 = \dfrac{\pi}{4}$

9. Given that $\cos^{-1}\dfrac{x}{a} + \cos^{-1}\dfrac{y}{b} = \theta$

$\Rightarrow \cos^{-1}\left[\dfrac{x}{a}.\dfrac{y}{b} - \sqrt{\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{b^2}\right)}\right] = \theta$

$\Rightarrow \dfrac{xy}{ab} - \sqrt{\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{b^2}\right)} = \cos\theta$

$\Rightarrow \dfrac{xy}{ab} - \cos\theta = \sqrt{\left(1-\dfrac{x^2}{a^2}\right)\left(1-\dfrac{y^2}{b^2}\right)}$

Squaring both sides we get,

$\dfrac{x^2y^2}{a^2b^2} - 2\dfrac{xy}{ab}\cos\theta + \cos^2\theta$

$= 1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{x^2y^2}{a^2b^2}$

$\Rightarrow \dfrac{x^2y^2}{a^2b^2} - 2\dfrac{xy}{ab}\cos\theta + \cos^2\theta$

$= 1 - \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} + \dfrac{x^2y^2}{a^2b^2}$

$\Rightarrow \dfrac{x^2}{a^2} - \dfrac{2xy}{ab}\cos\theta + \dfrac{y^2}{b^2} = 1 - \cos^2\theta$

$= \sin^2\theta \cdot$

10. Let $\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right) = \theta$

$\Rightarrow \tan\theta = \sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}$

$\cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-\dfrac{a-b}{a+b}\tan^2\dfrac{x}{2}}{1+\dfrac{a-b}{a+b}\tan^2\dfrac{x}{2}}$

$= \dfrac{a+b-(a-b)\tan^2\dfrac{x}{2}}{a+b+(a-b)\tan^2\dfrac{x}{2}}$

$= \dfrac{a\left(1-\tan^2\dfrac{x}{2}\right)+b\left(1+\tan^2\dfrac{x}{2}\right)}{a\left(1+\tan^2\dfrac{x}{2}\right)+b\left(1-\tan^2\dfrac{x}{2}\right)}$

$= \dfrac{a\dfrac{\left(1-\tan^2\dfrac{x}{2}\right)}{1+\tan^2\dfrac{x}{2}}+b}{a+b\left(\dfrac{1-\tan^2\dfrac{x}{2}}{1+\tan^2\dfrac{x}{2}}\right)}$

$= \dfrac{a\cos x + b}{a + b\cos x}$

$\Rightarrow \cos^{-1}\left(\dfrac{a\cos x + b}{a+b\cos x}\right) = 2\theta$

$= 2\tan^{-1}\left(\sqrt{\dfrac{a-b}{a+b}}\tan\dfrac{x}{2}\right)$

11. Let $\sin^{-1}\dfrac{x}{a} = \alpha,\ \sin^{-1}\dfrac{y}{b} = \beta$

$\Rightarrow \dfrac{x}{a} = \sin\alpha,\ \dfrac{y}{b} = \sin\beta$

$\therefore \quad \cos\alpha = \sqrt{1-\frac{x^2}{a^2}}, \cos\beta = \sqrt{1-\frac{y^2}{b^2}}$

Given that $\sin^{-1}\frac{x}{a} + \sin^{-1}\frac{y}{b} = \sin^{-1}\left(\frac{c^2}{ab}\right)$

$\Rightarrow \alpha + \beta = \sin^{-1}\left(\frac{c^2}{ab}\right)$

$\Rightarrow \cos(\alpha+\beta) = \cos\left(\sin^{-1}\frac{c^2}{ab}\right)$

$\Rightarrow \sqrt{\frac{1-x^2}{a^2}}\sqrt{\frac{1-y^2}{b^2}} - \frac{x}{a}.\frac{y}{b} = \sqrt{1-\frac{c^4}{a^2b^2}}$

$\sqrt{(a^2-x^2)(b^2-y^2)} - xy = \sqrt{a^2b^2-c^4}$

$\Rightarrow \sqrt{a^2b^2 - a^2y^2 - b^2x^2 + x^2y^2}$

$= xy + \sqrt{a^2b^2-c^4}$

Squaring both sides, we get

$a^2b^2 - a^2y^2 - b^2x^2 + x^2y^2$

$= x^2y^2 + a^2b^2 - c^4 + 2xy\sqrt{a^2b^2-c^4}$

$\Rightarrow b^2x^2 + 2xy\sqrt{a^2b^2-c^4} + a^2y^2 = c^4.$

12. Let $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

L.H.S = $\tan^{-1}\left[\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right]$

$= \tan^{-1}\left[\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right]$

$= \tan^{-1}\left(\frac{\sqrt{2\cos^2\frac{\theta}{2}}-\sqrt{2\sin^2\frac{\theta}{2}}}{\sqrt{2\cos^2\frac{\theta}{2}}+\sqrt{2\sin^2\frac{\theta}{2}}}\right)$

$= \tan^{-1}\left(\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right)$

$= \tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right)$

$= \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$

$= \frac{\pi}{4}-\frac{1}{2}\theta = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x.$

13. The given L.P.P is

Maximize $z = 4x_1 + 3x_2$ ....(1)

Subject to $x_1 + x_2 \le 50$ ....(2)

$x_1 + 2x_2 \le 80$ ....(3)

$2x_1 + x_2 \ge 20$ ....(4)

$x_1, x_2 \ge 0$ ......(5)

Changing the inequations to equations, we get

$x_1 + x_2$ ....(6)

$x_1 + 2x_2 = 80$ ....(7)

$2x_1 + x_2 = 20$ ....(8)

$x_1 = 0, x_2 = 0$ ....(9)

From equation (6), we get see that

x_1	0	50
x_2	50	0

The line (6) passes through (0, 50) and (50, 0)

Again putting (0, 0) in inequation (2), we get

$0+0 \le 50 \quad \Rightarrow 0 \quad \Rightarrow 50$ which is true.

The half plane is towards the origin.

From equation (7), we get

x_1	0	80
x_2	40	0

The line (7) passes through (0, 40) and (80, 0).

Putting (0, 0) in inequation (3), we get

$0+0.0 \le 80 \quad \Rightarrow \quad 0 \le 80$ which is true.

So the half plane is towards the origin.

From the equation (8), we get

x_1	0	10
x_2	20	0

The line (8) passes through (0, 20) and (10, 0).
Putting (0, 0) in inequation (4), we get

$2.0 + 0 \geq 20 \quad \Rightarrow \quad 0 \geq 20$ which is false.

So the half plane is towards the origin.

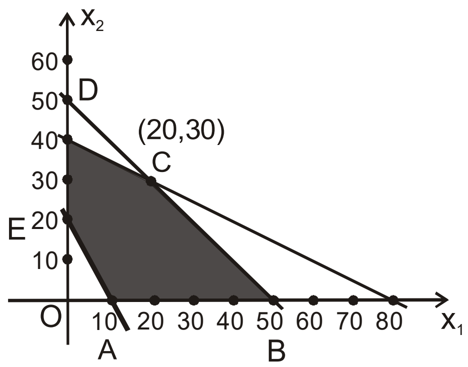
Solving (6) and (7), we get x_1 = 20, x_2 = 30.

The line (6) and (7) intersect at (20, 30).

The graph of L.P.P is as shown in the figure.

The feasible region is ABCDE where A (10,0), B(50,0), C(20,30), D(0,40), E(0,20).

The value of z is given in the following table



Point	x_1	x_2	$z = 4x_1 + 3x_2$
A	10	0	$z = 4 \times 10 + 3.0 = 40$
B	50	0	$z = 4 \times 50 + 3.0 = 200$
C	20	30	$z = 4 \times 20 + 3 \times 30 = 170$
D	0	40	$z = 4 \times 0 + 3 \times 40 = 120$
E	0	20	$z = 4 \times 0 + 3 \times 20 = 60$

The maximum value of z is 200 and it is obtained when x_1 = 50, x_2 = 0.

14. The given L.P.P is

Maximize $z = 20x + 10y$ ....(1)

Subject to $x + 2y \leq 40$ ....(2)

$3x + y \geq 30$ ....(3)

$4x + 3y \geq 60$ ....(4)

$x, y \geq 0$ ......(5)

Reducing the inequations to equations, we get

$x + 2y = 40$ ....(6)

$3x + y = 30$ ....(7)

$4x + 3y = 60$ ....(8)

$x = 0, y = 0$ ....(9)

From equation (6), we get see that

x	0	40
y	2	0

The line (6) passes through (0, 20) and (40, 0) putting (0, 0) in inequation (2), we have

$0 + 2.0 \leq 40 \quad \Rightarrow \quad 0 \quad \Rightarrow \quad 40$ which is true.

The half plane is towards the origin.
From equation (7), we get

x	0	10
y	30	0

The line (7) passes through (0, 30) & (10, 0).
Putting (0, 0) in inequation (3), we get

$3.0 + 0 \leq 30 \quad \Rightarrow \quad 0 \geq 30$ which is false.

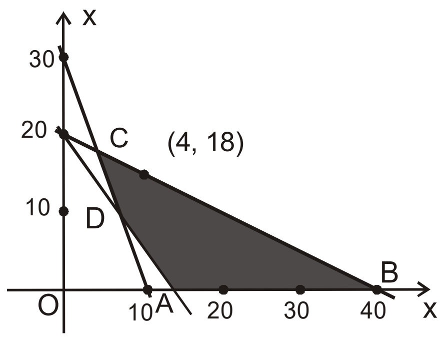
So the half plane is away from the origin.
From the equation (8), we get

x	0	15
y	20	0

The line (8) passes through (0, 20) and (15, 0).
Putting (0, 0) in (4), we get

$4.0 + 3.0 \geq 60 \quad \Rightarrow \quad 0 \geq 60$ which is false.

So the half plane is away from the origin.
Solving (6) and (7), we get x_1 = 20, x_2 = 30.
The graph of L.P.P is as shown in the figure.



Solving (6) and (7), we get

x = 4, y = 18.

The lines (6) & (7) intersect at (4, 18).

Solving (7) and (8), we get

x = 6, y = 12.

The lines (7) and (8) intersect at (6, 12).

The feasible region is ABCD where A (15,0), B(40,0), C(4,18), D(6,12).

The value of z at different points are given in the following table.

Point	x	y	$z = 20x + 10y$
A	15	0	$z = 20 \times 15 + 10 \times 0 = 300$
B	40	0	$z = 20 \times 40 + 10 \times 0 = 800$ (Max)
C	4	18	$z = 20 \times 4 + 10 \times 18 = 260$
D	6	12	$z = 20 \times 6 + 10 \times 12 = 240$

The maximum value of z is 800 and it is obtained when x = 40, y = 0.

15. The given L.P.P is

Maximize $z = 20x_1 + 40x_2$ ....(1)

Subject to $36x_1 + 6x_2 \geq 108$ ....(2)

$3x_1 + 12x_2 \geq 36$ ....(3)

$2x_1 + x_2 \geq 10$ ....(4)

$x_1, x_2 \geq 0$ ......(5)

Converting the inequations to equations, we get

$36x_1 + 6x_2 = 108$ ....(6)

$3x_1 + 12x_2 = 36$ ....(7)

$2x_1 + x_2 = 10$ ....(8)

$x_1 = 0, x_2 = 0$ ....(9)

To draw the line (6), we take the following data.

x_1	0	3
x_2	18	0

The line (6) passes through the points (0, 18) and (3, 0).

To draw the line (7), we take

x_1	0	12
x_2	3	0

The line (7) passes through the points (0, 3) and (12, 0).

To draw the line (8), we take

x_1	0	5
x_2	10	0

The line (8) passes through the points (0, 10) and (5, 0)

Putting (0, 0) in inequation (2), we get

$36.0 + 6.0 \geq 108$

$\Rightarrow \ 0 \geq 108$ which is false.

The half plane of (2) is away from the origin.

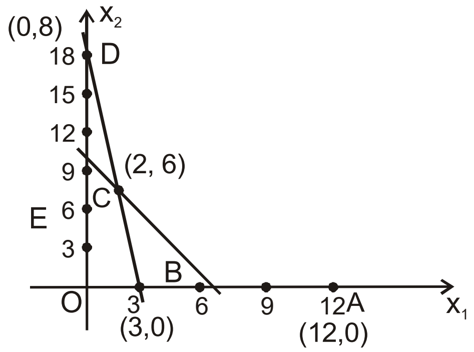
Putting (0, 0) in inequation (3), we have

$3.0 + 12.0 \geq 36$

$\Rightarrow \ 0 \geq 36$ which is false.

The half plane of (4) is away from the origin.

The graph of L.P.P is as shown in the figure.



x_1 ABCD x_2 is the feasible region. The vertices of the feasible region are A(12,0), B(4,2), C(2,6) and D (0,18).

The value of the objective function is given in the following table

Point	x_1	x_2	$z = 20x_1 + 40x_2$
A	12	0	$z = 20 \times 12 + 40 \times 0 = 240$
B	4	2	$z = 20 \times 4 + 40 \times 2 = 800$ (Min)
C	2	6	$z = 20 \times 2 + 40 \times 16 = 280$
D	0	18	$z = 20 \times 0 + 40 \times 18 = 720$

The maximum value of z is 160 and it is obtained when $x_1 = 4$, $y_1 = 2$.

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8. Matrices, Determinants, Probability. Each question carries 6 marks.

1. The given equations are

$$x-2y=3$$
$$3x+4y-z=-2$$
$$5x-3z=-1$$

$$\Rightarrow \quad x-2y+0z=3$$
$$3x+4y-z=-2$$
$$5x+0y-3z=-1$$

$$\Rightarrow \begin{bmatrix}1 & -2 & 0\\ 3 & 4 & -1\\ 5 & 0 & -3\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}3\\ -2\\ -1\end{bmatrix}$$

$$\Rightarrow \quad AX=B \qquad \text{..........(1)}$$

Where $A=\begin{bmatrix}1 & -2 & 0\\ 3 & 4 & -1\\ 5 & 0 & -3\end{bmatrix}, X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}, B=\begin{bmatrix}3\\ -2\\ -1\end{bmatrix}$

$$|A|=\begin{vmatrix}1 & -2 & 0\\ 3 & 4 & -1\\ 5 & 0 & -3\end{vmatrix}$$

$= 1(-12.0)-(-2)(-9+5)+0$

$= -12+(-8)=-20 \neq 0$

The given system of equations are consistent and solvable.

From (1), we have

$$X=A^{-1}B$$

$$\Rightarrow \quad X=\frac{1}{[A]}(\operatorname{adj}A)B \qquad \text{.......... (2)}$$

A_{11} = cofactor of $1=(-1)^{1+1}\begin{vmatrix}4 & -1\\ 0 & -3\end{vmatrix}=-12$

$A_{12}=4,\ A_{13}=-20$

$A_{21}=-6,\ A_{22}=-3,\ A_{23}=-10$

$A_{31}=2\ \ A_{32}=1\ \ A_{33}=10$

Matrix of the cofactors = $\begin{bmatrix}-12 & 4 & -20\\ -6 & -3 & -10\\ 2 & 1 & 10\end{bmatrix}$

$$\operatorname{Adj} A=\begin{bmatrix}-12 & -6 & 2\\ 4 & -3 & 1\\ -20 & -10 & 10\end{bmatrix}$$

From (2), we get

$$\lambda=-\frac{1}{20}\begin{bmatrix}-12 & -6 & 2\\ 4 & -3 & 1\\ -20 & -10 & 10\end{bmatrix}\begin{bmatrix}3\\ -2\\ -1\end{bmatrix}$$

$$=-\frac{1}{20}\begin{bmatrix}-36 & +12 & -2\\ 12 & +6 & -1\\ -60 & +20 & -10\end{bmatrix}$$

$$\Rightarrow \begin{bmatrix}x\\ y\\ z\end{bmatrix}=-\frac{1}{20}\begin{bmatrix}-26\\ 17\\ -50\end{bmatrix}=\begin{bmatrix}\frac{26}{20}\\ -\frac{17}{20}\\ \frac{50}{20}\end{bmatrix}$$

$$\Rightarrow \quad x=\frac{26}{20},\ y=-\frac{17}{20},\ z=\frac{50}{20}$$

2. Let $A=\begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 4\\ 1 & 0 & 2\end{bmatrix}$

We know A = IA

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 4\\ 1 & 0 & 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A.$$

$$\Rightarrow \begin{bmatrix}1 & 2 & 3\\ 0 & -3 & -2\\ 0 & -2 & -1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -2 & 1 & 0\\ -1 & 0 & 1\end{bmatrix}A$$

$[R_2 \rightarrow R_2-2R_1$
$R_3 \rightarrow R_3-R_1]$

$$\Rightarrow \begin{bmatrix} 3 & 6 & 9 \\ 0 & -3 & -2 \\ 0 & -6 & -3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 3 \end{bmatrix} A$$

$[R_1 \to 3R_1$
$R_3 \to 3R_3]$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 5 \\ 0 & -3 & -2 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & 1 & 0 \\ 1 & -2 & 3 \end{bmatrix} A$$

$[R_1 \to R_1 + 2R_2$
$R_3 \to R_3 - 2R_2]$

$$\Rightarrow \begin{bmatrix} 3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -6 & 12 & -15 \\ 0 & -3 & 6 \\ 1 & -2 & 3 \end{bmatrix} A$$

$[R_1 \to R_1 - 5R_3$
$R_3 \to R_2 + 2R_3]$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix} A$$

$[R_1 \to \frac{1}{3}R_1$
$R_2 \to -\frac{1}{3}R_2]$

Thus, $A^{-1} = \begin{bmatrix} -2 & 4 & -5 \\ 0 & 1 & -2 \\ 1 & -2 & 3 \end{bmatrix}$

3. Given that $A = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 6 & -3 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{bmatrix}$

$$\therefore \quad AB = \begin{bmatrix} 1 & 2 & 3 \\ 6 & 7 & 8 \\ 6 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 2 \\ 5 & 6 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+6+15 & 2+8+18 & 3+4+3 \\ 6+21+40 & 12+28+48 & 18+14+8 \\ 6-9+20 & 12-12+24 & 18-6+4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 28 & 10 \\ 67 & 88 & 40 \\ 17 & 24 & 16 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 22 & 67 & 17 \\ 28 & 88 & 24 \\ 10 & 40 & 16 \end{bmatrix} \quad ....(1)$$

$$A^T = \begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & 8 & 4 \end{bmatrix}, B^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix}$$

$$B^TA^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 6 & 6 \\ 2 & 7 & -3 \\ 3 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 22 & 67 & 17 \\ 28 & 88 & 24 \\ 10 & 40 & 16 \end{bmatrix} \quad ....(2)$$

From (1) and (2), we have

$$(AB)^T = B^TA^T.$$

4. $A = \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{bmatrix}$

$$\therefore \quad |A| = \begin{vmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{vmatrix}$$

$= 3(1+6)-(-2)(2-3)+1(4+1)$
$= 21-2+5 = 24 \neq 0.$

A_{11} = cofactor of 3 = $(-1)^{1+1} \begin{vmatrix} 1 & -3 \\ 2 & 1 \end{vmatrix} = 1-(-6) = 7$

$A_{12} = (-1)^{1+2} \begin{vmatrix} 2 & -3 \\ -1 & 1 \end{vmatrix} = -(2-3) = 1$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} 2 & 1 \\ -1 & 2 \end{vmatrix} = 4+1=5$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} -2 & 1 \\ 2 & 1 \end{vmatrix} = -(-2-2)=4$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 3 & 1 \\ -1 & 1 \end{vmatrix} = 3-(-1)=4$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 3 & -2 \\ -1 & 2 \end{vmatrix} = -(6-2)=-4$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} -2 & 1 \\ 1 & -3 \end{vmatrix} = 6-1=5$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 3 & 1 \\ 2 & -3 \end{vmatrix} = -(-9-2)=11$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 3 & -2 \\ 2 & 1 \end{vmatrix} = 3+4=7$$

Matrix of cofactors = $\begin{bmatrix} 7 & 1 & 5 \\ 4 & 4 & -4 \\ 5 & 11 & 7 \end{bmatrix}$

$$\text{Adj } A = \begin{bmatrix} 7 & 4 & 5 \\ 1 & 4 & 11 \\ 5 & -4 & 7 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|}.\text{adj } A$$

$$= \frac{1}{24}\begin{bmatrix} 7 & 4 & 5 \\ 1 & 4 & 11 \\ 5 & -4 & 7 \end{bmatrix}$$

Given equations are

$3x - 2y + z = 2$

$2x + y - 3z = -5$

$-x + 2y + z = 6$

$$\Rightarrow \begin{bmatrix} 3 & -2 & 1 \\ 2 & 1 & -3 \\ -1 & 2 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$$

$\Rightarrow AX = B$

$\Rightarrow X = A^{-1}B$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 7 & 4 & 5 \\ 1 & 4 & 11 \\ 5 & -4 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ -5 \\ 6 \end{bmatrix}$$

$$= \frac{1}{24}\begin{bmatrix} 14-20+30 \\ 2-20-66 \\ 10+20+42 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 24 \\ 48 \\ 72 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore$ x, 1, y=2, z = 3.

5. Given that $A + 2B = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix}$ ....(1)

$$2A - B = \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix} \quad ......(2)$$

Multiplying 2 in (2), we get

$$4A - 2B = \begin{bmatrix} 4 & -2 & 10 \\ 4 & -2 & 12 \\ 0 & 2 & 4 \end{bmatrix} \quad .....(3)$$

Adding (1) and (3), we get

$$5A = \begin{bmatrix} 1 & 2 & 0 \\ 6 & -3 & 3 \\ -5 & 3 & 1 \end{bmatrix} + \begin{bmatrix} 4 & -2 & 10 \\ 4 & -2 & 12 \\ 0 & 2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 0 & 10 \\ 10 & -5 & 15 \\ -5 & 5 & 5 \end{bmatrix}$$

$$\Rightarrow A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{bmatrix}$$

From (2), we get

$$B = 2A - \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 4 \\ 4 & -2 & 6 \\ -2 & 2 & 2 \end{bmatrix} - \begin{bmatrix} 2 & -1 & 5 \\ 2 & -1 & 6 \\ 0 & 1 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & -1 \\ 2 & -1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$$

6. $\text{L.H.S} = \begin{vmatrix} (b+c)^2 & a^2 & bc \\ (c+a)^2 & b^2 & ca \\ (a+b)^2 & c^2 & ab \end{vmatrix}$

$$= \begin{vmatrix} (b+c)^2 - 2bc & a^2 & bc \\ (c+a)^2 - 2ca & b^2 & ca \\ (a+b)^2 - 2ab & c^2 & ab \end{vmatrix} \quad (c_1 \to c_1 - 2c_3)$$

$$= \begin{vmatrix} b^2+c^2 & a^2 & bc \\ c^2+a^2 & b^2 & ca \\ a^2+b^2 & c^2 & ab \end{vmatrix}$$

$$= \begin{vmatrix} a^2+b^2+c^2 & a^2 & bc \\ a^2+b^2+c^2 & b^2 & ca \\ a^2+b^2+c^2 & c^2 & ab \end{vmatrix} \quad (c_1 \to c_1 + c_2)$$

$$= \left(a^2+b^2+c^2\right) \begin{vmatrix} 1 & a^2 & bc \\ 1 & b^2 & ca \\ 1 & c^2 & ab \end{vmatrix}$$

$$= \left(a^2+b^2+c^2\right) \begin{vmatrix} 1 & a^2 & bc \\ 0 & b^2-a^2 & ca-bc \\ 0 & c^2-a^2 & ab-bc \end{vmatrix}$$

$[R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1]$

$$= \left(a^2+b^2+c^2\right) \begin{vmatrix} b^2-a^2 & ca-bc \\ c^2-a^2 & ab-bc \end{vmatrix}$$

$$= \left(a^2+b^2+c^2\right) \begin{vmatrix} -(a-b)(a+b) & c(a+b) \\ (c-a)(c+a) & -b(c-a) \end{vmatrix}$$

$$= \left(a^2+b^2+c^2\right)(a-b)(c-a) \begin{vmatrix} -(a+b) & c \\ c+a & -b \end{vmatrix}$$

$$= \left(a^2+b^2+c^2\right)(a+b+c)(a-b)(b-c)(c-a)$$

7. Let $\Delta = \begin{vmatrix} 1+a^2-b^2 & 2ab & -2b \\ 2ab & 1-a^2+b^2 & 2a \\ 2b & -2a & 1-a^2-b^2 \end{vmatrix}$

On applying $c_1 \to c_1 - bc_3$ & $c_2 \to c_2 + ac_3$, we get

$$\Delta = \begin{vmatrix} 1+a^2+b^2 & 0 & -2b \\ 0 & 1+a^2+b^2 & 2a \\ b\left(1+a^2+b^2\right) & a\left(1+a^2+b^2\right) & 1-a^2-b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \begin{vmatrix} 1 & 0 & -2b \\ 0 & 1 & 2a \\ b & -a & 1-a^2-b^2 \end{vmatrix}$$

$$= \left(1+a^2+b^2\right)^2 \left[\left(1-a^2-b^2\right) + 2a^2 - 0 + (2b)(0-b)\right)$$

$$= \left(1+a^2+b^2\right)^2 \left[\left(1-a^2-b^2+2a^2+2b^2\right)\right.$$

$$= \left(1+a^2+b^2\right)^2 \left(1+a^2+b^2\right)$$

$$= \left(1+a^2+b^2\right)^3$$

8. Let $\Delta = \begin{vmatrix} a & b & c \\ a-b & b-c & c-a \\ b+c & c+a & a+b \end{vmatrix}$

$$= \begin{vmatrix} a+b+c & b & c \\ 0 & b-c & c-a \\ 2(a+b+c) & c+a & a+b \end{vmatrix} \quad (c_1 \to c_1 + c_2 + c_3)$$

$$= (a+b+c)\begin{vmatrix}1 & b & c\\0 & b-c & c-a\\2 & c+a & a+b\end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix}1 & b & c\\0 & b-c & c-a\\0 & c+a-2b & a+b-2c\end{vmatrix}$$

$(R_3 \to R_3 - 2R_1)$

$$= (a+b+c)\begin{vmatrix}b-c & c-a\\c+a-2b & a+b-2c\end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix}b-c & c-a\\a-c & b-a\end{vmatrix} \quad (R_2 \to R_2 + 2R_1)$$

$$= (a+b+c)\left[(b-c)(b-a)-(a-c)(c-a)\right]$$

$$= (a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)$$

$= a^3 + b^3 + c^3 - 3abc.$

9. Given that $x \neq y \neq z$.

Given that $\Delta = \begin{vmatrix}x & x^2 & 1+x^3\\y & y^2 & 1+y^3\\z & z^2 & 1+z^3\end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix}x & x^2 & 1\\y & y^2 & 1\\z & z^2 & 1\end{vmatrix} + \begin{vmatrix}x & x^2 & x^3\\y & y^2 & y^3\\z & z^2 & z^3\end{vmatrix} = 0$$

$$\Rightarrow (-1)(-1)\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix} + xyz\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}(1+xyz) = 0$$

$$\begin{vmatrix}1 & x & x^2\\1 & y & y^2\\1 & z & z^2\end{vmatrix}$$

$$= \begin{vmatrix}1 & x & x^2\\0 & y-x & y^2-x^2\\0 & z-x & z^2-x^2\end{vmatrix}$$

$$= \begin{vmatrix}y-x & y^2-x^2\\z-x & z^2-x^2\end{vmatrix}$$

$$= \begin{vmatrix}y-x & (y-x)(y+x)\\z-x & (z-x)(z+x)\end{vmatrix}$$

$$= (b-x)(z-x)\begin{vmatrix}1 & y+x\\1 & z+x\end{vmatrix}$$

$= (x-y)\ (y-z)\ (z-x)$

From (1), we have

$(x-y)\ (y-z)\ (z-x)\ (1+xyz)=0$

$\Rightarrow 1 + xy\,z = 0 \quad (\because x \neq y \neq z)$

$\Rightarrow xyz = -1$

10. Let $\Delta = \begin{vmatrix}2y & y-z-x & 2y\\2z & 2z & 2-x-y\\x-y-z & 2x & 2x\end{vmatrix}$

$$= \begin{vmatrix}x+y+z & x+y+z & x+y+z\\2z & 2z & 2-x-y\\x-y-z & 2x & 2x\end{vmatrix}$$

$(R_1 \to R_1 + R_2 + R_3)$

$$= (x+y+z)\begin{vmatrix}1 & 1 & 1\\2z & 2z & z-x-y\\x-y-z & 2x & 2x\end{vmatrix}$$

$$= (x+y+z)\begin{vmatrix}1 & 0 & 0\\2z & 0 & -(x+y+z)\\x-y-z & x+y+z & x+y+z\end{vmatrix}$$

$$= (x+y+z)^3\begin{vmatrix}1 & 0 & 0\\2z & 0 & -1\\x-y-z & 1 & 1\end{vmatrix}$$

$$= (x+y+z)^3 - 1 = (x+y+z)^3$$

11. Let E_1, E_2 and E_3 be the events of selecting the bag B_1, B_2 and B_3 respectively.

$$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

Let A be the event of selecting the white balls.

$$P(A|E_1) = \frac{4}{9}$$

$$P(A|E_2) = \frac{3}{8}$$

$$P(A|E_3) = \frac{5}{7}$$

By the theorem of total probability P (white ball) = P(A)

$$= P(E_1).P(A|E_1) + P(E_2).P(A|E_2) + P(E_3).P(A|E_3)$$

$$= \frac{1}{3}.\frac{4}{9} + \frac{1}{3}.\frac{3}{8} + \frac{1}{3}.\frac{5}{7}$$

$$= \frac{1}{3}\left(\frac{4}{9} + \frac{3}{8} + \frac{5}{7}\right)$$

$$= \frac{1}{3}\ \frac{224 + 189 + 360}{504}$$

$$= \frac{773}{1512}$$

12. Let X be the random variable which represents the number of doublets in 4 throws of a pair of dice.
X can take the values 0, 1, 2, 3, 4.
The possible doublets are (1,1), (2,2), (3,3), (4,4), (5,5) and (6,6)
Probability of getting a doublet

$$= P(\text{doublet}) = \frac{6}{36} = \frac{1}{6}.$$

P (non doublet) = $1 - \frac{1}{6} = \frac{5}{6}$.

The given experiment is a binomial distribution with n = 4.

$$P = \frac{1}{6},\ q = \frac{5}{6}.$$

$$P(X=0) = {}^4c_0\, p^0 q^4 = 1.\left(\frac{5}{6}\right)^4 = \frac{625}{1296}$$

$$P(X=1) = {}^4c_1\, p^1 q^3 = 4.\frac{1}{6}.\left(\frac{5}{6}\right)^3 = \frac{500}{1296}$$

$$P(X=2) = {}^4c_2\, p^2 q^2 = 6.\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^2 = \frac{150}{1296}$$

$$P(X=3) = {}^4c_3\, p^3 q^1 = 4\left(\frac{1}{6}\right)^3.\frac{5}{6} = \frac{20}{1296}$$

$$P(X=4) = {}^4c_4\, p^4 q^0 = 1.\left(\frac{1}{6}\right)^4 = \frac{1}{1296}$$

The required probability distribution is

$X = x$	0	1	2	3	4
p(x)	$\frac{625}{1296}$	$\frac{500}{1296}$	$\frac{150}{1296}$	$\frac{20}{1296}$	$\frac{1}{12}$

As it is a binomial distribution,

its mean = $np = 4.\frac{1}{6} = \frac{4}{6} = \frac{2}{3}$

Variance = $npq = 4.\frac{1}{6}.\frac{5}{6} = \frac{5}{9}$.

13. In a single throw of a pair dice if S is the sample space, then |S| = 36.
The event of getting a doublets

$$= \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$$

P (a doublet) = $\frac{6}{36} = \frac{1}{6}$

P (a non doublet) = $1 - \frac{1}{6} = \frac{5}{6}$.

Here n = 5, $p = \frac{1}{6}$, $q = \frac{5}{6}$.

P (3 doublet in 5 throw)

$$= {}^5c_3 p^3 q^2$$

$$= 10.\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2$$

$$= \frac{250}{65} = \frac{125}{3888}$$

14. Let be the random variable of the number of defective bulbs.

x takes values o or 1 or 2 or 3 or 4.

Number of non-defective bulbs in the box = 20 – 5 = 15.

Probability of getting a non defective bulbs

$= \frac{15}{20} = \frac{3}{4}.$

Hence the probability of getting a detective

buld $= 1 - \frac{3}{4} = \frac{1}{4}$

$P(x=0) = \frac{1}{4}.\frac{1}{4}.\frac{1}{4}.\frac{1}{4} = \frac{1}{256}$

$P(x=1) = \frac{3}{4}.\frac{1}{4}.\frac{1}{4}.\frac{1}{4}.\frac{1}{4}.\frac{3}{4}.\frac{1}{4}.\frac{1}{4}$

$+\frac{1}{4}.\frac{1}{4}.\frac{3}{4}.\frac{1}{4} + \frac{1}{4}\frac{1}{4}\frac{1}{4}\frac{3}{4}$

$= 4.\frac{3}{4}.\frac{1}{4}.\frac{1}{4}.\frac{1}{4} = \frac{12}{256}$

$P(x=2) = \frac{3}{4}.\frac{3}{4}.\frac{1}{4}.\frac{1}{4} + \frac{1}{4}.\frac{3}{4}.\frac{3}{4}.\frac{1}{4} + \frac{1}{4}.\frac{1}{4}.\frac{3}{4}.\frac{3}{4}$

$+\frac{3}{4}.\frac{1}{4}.\frac{3}{4}.\frac{1}{4} + \frac{3}{4}.\frac{1}{4}.\frac{1}{4}\frac{3}{4} + \frac{1}{4}.\frac{3}{4}.\frac{1}{4}.\frac{3}{4}$

$= 6.\frac{3}{4}.\frac{3}{4}.\frac{1}{4}.\frac{1}{4} = \frac{54}{256}$

$P(x=3) = \frac{3}{4}.\frac{3}{4}.\frac{3}{4}.\frac{1}{4} + \frac{3}{4}.\frac{3}{4}.\frac{1}{4}.\frac{3}{4}$

$+\frac{3}{4}.\frac{1}{4}.\frac{3}{4}.\frac{3}{4} + \frac{1}{4}.\frac{3}{4}.\frac{3}{4}.\frac{3}{4}$

$= 4.\frac{3}{4}.\frac{3}{4}.\frac{3}{4}.\frac{1}{4} = \frac{108}{256}.$

$P(x=4) = \frac{3}{4}.\frac{3}{4}.\frac{3}{4}.\frac{3}{4} = \frac{81}{256}$

Thus the required probability distribution y

$X = x$	0	1	2	3	4
$p(x)$	$\frac{1}{256}$	$\frac{12}{256}$	$\frac{54}{256}$	$\frac{108}{256}$	$\frac{81}{256}$

Mean $\overline{x} = \sum_{i=1}^{5} x_i p(x_i)$

$= 0.\frac{1}{256} + 1.\frac{12}{256} + 2.\frac{54}{256} + 3.\frac{108}{256} + 4.\frac{81}{256}$

$= 0.\frac{12}{256} + \frac{108}{256} + \frac{324}{256} + \frac{324}{256}$

$= \frac{768}{256} = 3.$

Variance $= \sum_{i=1}^{5} x_i^2 p(x_i) - x^{-2}$

$= 0^2.\frac{1}{256} + 1^2\frac{12}{256} + 2^2.\frac{54}{256} + 3^2.\frac{108}{256}$

$+ 4^2.\frac{81}{256} - 3^2$

$= \frac{12}{256} + \frac{54}{256} + \frac{972}{256} + \frac{1296}{256} - 9$

$= \frac{2496}{256} - 9 = \frac{3}{4}.$

15. 5 boys and 4 girls randomly stand in a line. These 9 boys and girls aramaped among themselves is 9! ways

|S| = 9!

They are arranged among themselves such that no two girls come together.

For 4 girls, there 6 positions. (for in between the boys and two in each exteme). 4 girls can stand in 6 position in $^6p_4$ ways.

Again 5 boys can stand in 5! ways.

Probability that they all stand in a line

$= \frac{5! \times {}^6p_4}{9!} = \frac{5}{42}$

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9. Continuity and differentiability, Applications of derivatives. Each question carries 6 marks.

1. Given that $e^{\frac{y}{x}} = \frac{x}{a+bx}$

$$\Rightarrow \frac{y}{x} = \ln\frac{x}{a+bx} \quad \text{........(1)}$$

$$\Rightarrow y = x\ln\frac{x}{a+bx}$$

$$= x\left[\ln x - \ln(a+bx)\right]$$

$$\frac{dy}{dx} = \ln x - \ln(a+bx) + x\left[\frac{1}{x} - \frac{b}{a+bx}\right]$$

$$= \ln\frac{x}{a+bx} + 1 - \frac{bx}{a+bx}$$

$$= \frac{y}{x} + 1 - \frac{bx}{a+bx} \quad \text{........(2)}$$

$$\Rightarrow x\frac{dy}{dx} = y + x - \frac{bx^2}{a+bx}$$

$$\Rightarrow x\frac{dy}{dx} - y = x\left(1 - \frac{bx}{a+bx}\right)$$

$$= x\left(\frac{a+bx-bx}{a+bx}\right)$$

$$= \frac{ax}{a+bx}$$

$$\Rightarrow \left(x\frac{dy}{dx} - y\right)^2 = \frac{a^2x^2}{(a+bx)^2} \quad \text{........(3)}$$

Again from (2), we have

$$\frac{dy}{dx} = \ln x - \ln(a+bx) + \frac{a}{a+bx}$$

$$\therefore \quad \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{x} - \frac{b}{a+bx} - \frac{a}{(a+bx)^2}.b$$

$$= \frac{a+bx-bx}{x(a+bx)} - \frac{ab}{(a+bx)^2}$$

$$= \frac{a}{x(a+bx)} - \frac{ab}{(a+bx)^2}$$

$$= \frac{a}{a+bx}\left[\frac{1}{x} - \frac{b}{a+bx}\right]$$

$$= \frac{a}{a+bx}\cdot\frac{a+bx-bx}{x(a+bx)}$$

$$= \frac{a^2}{x(a+bx)^2}$$

$$\Rightarrow x^3\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{a^2x^2}{(a+bx)^2} \quad \text{.......(4)}$$

From (3) and (4), we get

$$x^3\frac{d}{dx}\left(\frac{dy}{dx}\right) = \left(x\frac{dy}{dx} - y\right)^2$$

2. Given that

$$f(x) = \begin{cases} \frac{\sqrt{1+kx} - \sqrt{1-kx}}{x} & \text{when } 1 < x < 0 \\ \frac{2x+1}{x-1} & \text{when} \, o \le x < 1 \end{cases}$$

$$\text{L.H.L}: \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{\sqrt{1+kx} - \sqrt{1-kx}}{x}$$

$$= \lim_{x\to 0^-}\frac{\left(\sqrt{1+kx} - \sqrt{1-kx}\right)\left(\sqrt{1+kx} + \sqrt{1-kx}\right)}{x\left(\sqrt{1+kx} + \sqrt{-kx}\right)}$$

$$= \lim_{x\to 0^-}\frac{1+kx-(1-kx)}{x\left(\sqrt{1+kx} + \sqrt{1-kx}\right)}$$

$$= \lim_{x\to 0^-}\frac{2kx}{x\left(\sqrt{1+kx} + \sqrt{1-kx}\right)}$$

$$= \lim_{x\to 0^-}\frac{2k}{\sqrt{1+kx} + \sqrt{1-kx}} = \frac{2k}{2} = k.$$

R.H.L = $\lim_{x\to 0^+} f(x)$

$= \lim_{x\to 0^+} \frac{2x+1}{x-1} = \frac{0+1}{x-1} = -1$

Also $f(0) = \frac{2.0+1}{0-1} = -1$

Since the function is continuous at x = 0,

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0+} f(x) = f(0)$

$\Rightarrow$ k = –1 = –1 $\Rightarrow$ k = –1

3. Given that $x = \frac{1-\cos^2\theta}{\cos\theta} - \sec\theta - \cos\theta$

$$\frac{dx}{d\theta} = \frac{d(\sec\theta - \cos\theta)}{d\theta}$$

$= \sec\theta\tan\theta - (-\sin\theta)$

$= \sec\theta\tan\theta + \tan\theta.\cos\theta$

$= \tan\theta(\sec\theta + \cos\theta)$ ....(1)

Again $y = \frac{1-\cos^{2n}\theta}{\cos^n\theta} = \frac{1}{\cos^n\theta} - \cos^n\theta$

$= \sec^n\theta - \cos^n\theta$

$$\frac{dy}{dx} = n.\sec^{n-1}\theta.\sec\theta.\tan\theta - n\cos^{n-1}\theta(-\sin\theta)$$

$= n\sec^n\theta.\tan\theta + n\cos^n\theta.\tan\theta$

$= n\tan\theta(\sec^n\theta + \cos^n\theta)$ ......(2)

$$\frac{dy}{dx} = \frac{dy}{d\theta}.\frac{d\theta}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}}$$ ....(3)

$$= \frac{n\tan\theta(\sec^n\theta + \cos^n\theta)}{\tan\theta(\sec\theta + \cos\theta)}.$$

$$\left(\frac{dy}{dx}\right)^2 = n^2\frac{(\sec^n\theta + \cos^n\theta)}{(\sec\theta + \cos\theta)^2}$$ ....(4)

R.H.S = $n^2\left(\frac{y^2+4}{x^2+4}\right)$

$$= n^2\left[\frac{(\sec^n\theta - \cos^n\theta)^2 + 4}{(\sec\theta - \cos\theta)^2 + 4}\right]$$

$$= n^2\frac{(\sec^n\theta + \cos^n\theta)^2}{(\sec\theta + \cos\theta)^2}$$ ......(5)

From (y) and (5), we have

$$\left(\frac{dy}{dx}\right)^2 = n^2\left(\frac{y^2+4}{x^2+4}\right)$$

4. Given that $y = \cot^{-1}(\ln\cos e^{-x}) + \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$

Let $u = \cot^{-1}(\ln\cos e^{-x})$, $v = \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$

$\therefore$ y = u + v

$\therefore$ $\frac{dy}{dx} = \frac{dy}{dx} + \frac{dv}{dx}$ ......(1)

Now $u = \cot^{-1}(\ln\cos e^{-x})$

$$\frac{dy}{dx} = \frac{d\cot^{-1}(\ln\cos e^{-x})}{dx}$$

$$= \frac{d\cot^{-1}(\ln\cos e^{-x})}{d(\ln\cos e^{-x})}.\frac{d(\ln\cos e^{-x})}{d(\cos e^{-x})}.\frac{d\cos e^{-x}}{de^{-x}}.\frac{de^{-x}}{dx}$$

$$= -\frac{1}{1+\ln^2(\cos^{-x})}.\frac{1}{\cos e^{-x}}.-\sin e^{-x}.x - e^{-x}$$

$$= -\frac{e^{-x}\sin e^{-x}}{\cos e^{-x}\left[1+\ln^2(\cos e^{-x})\right]}$$ ......(2)

$$v = \frac{x\sin^{-1}x}{\sqrt{1-x^2}}$$

$$\frac{dv}{dx} = \frac{d}{dx}\left[\frac{x\sin^{-1}x}{\sqrt{1-x^2}}\right]$$

$$= \frac{\sqrt{1-x^2}.\frac{d\left(x\sin^{-1}x\right)}{dx} - x\sin^{-1}x.\frac{d\sqrt{1-x^2}}{dx}}{1-x^2}$$

$$\sqrt{1-x^2}.\left[x\frac{d\sin^{-1}x}{dx} + \sin^{-1}x.\frac{dy}{dx}\right]$$

$$= \frac{-x\sin^{-1}x.\frac{d\left(1-x^2\right)^{\frac{1}{2}}}{d\left(1-x^2\right)}.d\frac{\left(1-x^2\right)}{dx}}{1-x^2}$$

$$= \frac{\sqrt{1-x^2}.\left[\frac{x}{\sqrt{1-x^2}} + \sin^{-1}x\right] - x\sin^{-1}x\frac{1}{2\sqrt{1-x^2}}(-2x)}{1-x^2}$$

$$= \frac{x + \sqrt{1-x^2}\sin^{-1}x + \frac{x^2\sin^{-1}x}{\sqrt{1-x^2}}}{1-x^2}$$

$$= \frac{x\sqrt{1-x^2} + \left(1-x^2\right)\sin^{-1}x + x^2\sin^{-1}x}{\left(1-x^2\right)\sqrt{1-x^2}}$$

$$= \frac{x\sqrt{1-x^2} + \sin^{-1}x}{\left(1-x^2\right)\sqrt{1-x^2}} \qquad ..(3)$$

From (1), (2) & (3), we get

$$\frac{dy}{dx} = -\frac{e^{-x}\sin\left(e^{-x}\right)}{\cos\left(e^{-x}\right)\left[1+\ln^2\cos\left(e^{-x}\right)\right]} + \frac{x\sqrt{1-x^2} + \sin^{-1}x}{\left(1-x^2\right)\sqrt{1-x^2}}.$$

5. Given that $x^y = y^x + \tan^{-1}\frac{\cos x}{1+\sin x}$ ......(1)

Let $u = x^y$, $v = y^x$ and $w = \tan^{-1}\frac{\cos x}{1+\sin x}$

From (1), we get

$u = v + w$

$$\therefore \frac{dy}{dx} = \frac{dv}{dx} + \frac{dw}{dx} \qquad ......(2)$$

$u = x^y$

$\Rightarrow \ln u = \ln x^y = y \ln x$

Differentiating both sides w.r.t x, we get

$$\frac{d\ln u}{dx} = \frac{dy\ln x}{dx}$$

$$\Rightarrow \frac{1}{u}.\frac{du}{dx} = y.\frac{d\ln x}{dx} + \ln x.\frac{dy}{dx}$$

$$= \frac{y}{x} + \ln x.\frac{dy}{dx}$$

$$\Rightarrow \frac{du}{dx} = u\left(\frac{y}{x} + \ln x.\frac{dy}{dx}\right)$$

$$= x^y\left(\frac{y}{x} + \ln x.\frac{dy}{dx}\right)$$

$$= y\,x^{y-1} + x^y\ln x.\frac{dy}{dx} \qquad ...(3)$$

Again $v = y^x$

$\Rightarrow \ln v = \ln y^x = x \ln y$.

Differentiating both sides w.r.t x, we get

$$\frac{d\ln v}{dx} = \frac{dx\ln y}{dx}$$

$$\Rightarrow \frac{1}{v}.\frac{dv}{dx} = x\frac{d\ln y}{dy}.\frac{dy}{dx} + \ln y.\frac{dx}{dx}$$

$$= y^x\left(\frac{x}{y}\frac{dy}{dx} + \ln y\right)$$

$$= x\,y^{x-1}\frac{dy}{dx} + y^x\ln y \qquad .....(4)$$

Again $w = \tan^{-1}\left(\frac{\cos x}{1+\sin x}\right)$

$$= \tan^{-1}\left(\frac{\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}}{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}\right)$$

$$= \tan^{-1}\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}+2\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\frac{\left(1-\tan\frac{x}{2}\right)\left(1+\tan\frac{x}{2}\right)}{\left(1+\tan\frac{x}{2}\right)^2}$$

$$= \tan^{-1}\left(\frac{1-\tan\frac{x}{2}}{1+\tan\frac{x}{2}}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{x}{2}\right)=\frac{\pi}{4}-\frac{1}{2}x$$

$$\frac{dw}{dx}=\frac{d\left(\frac{\pi}{4}-\frac{1}{2}x\right)}{dx}$$

From (2), (3), (4) & (5), we get

$$yx^{y-1}+x^y\ln x\frac{dy}{dx}=xy^{x-1}\frac{dy}{dx}+y^x\ln y-\frac{1}{2}$$

$$\Rightarrow \left(x^y\ln x-xy^{x-1}\right)\frac{dy}{dx}=y^x\ln y-yx^{y-1}-\frac{1}{2}$$

$$\Rightarrow \frac{dy}{dx}=\frac{y^x\ln y-yx^{y-1}-\frac{1}{2}}{x^y\ln x-x\,y^{x-1}}$$

6. Let $y=\tan^{-1}\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}$ & $z=\ln\left(\frac{1+\cos x}{1-\cos x}\right)$

We shall find out $\frac{dy}{dz}$.

$$\therefore \frac{dy}{dz}=\frac{\frac{dy}{dx}}{\frac{dz}{dx}} \qquad ......(1)$$

$$\frac{dy}{dx}=\frac{d\tan^{-1}\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}}{dx}$$

$$=\frac{d\tan^{-1}\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}}{d\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}}.\frac{d\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}}{d\left(\frac{1+\sin x}{1-\sin x}\right)}.d\frac{\left(\frac{1+\sin x}{1-\sin x}\right)}{dx}$$

$$=\frac{1}{1+\frac{1+\sin x}{1-\sin x}}\times\frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{-\frac{1}{2}}.$$

$$\frac{(1-\sin x)d\frac{(1+\sin x)}{dx}-(1+\sin x)d\frac{(1-\sin x)}{dx}}{(1-\sin x)^2}$$

$$=\frac{1-\sin x}{2}\times\frac{1}{2}\sqrt{\frac{1-\sin x}{1+\sin x}}.$$

$$\frac{(1-\sin x)\cos x-(1+\sin x)(-\cos x)}{(1-\sin x)^2}$$

$$=\frac{1-\sin x}{2}\times\frac{1}{2}\sqrt{\frac{1-\sin x}{1+\sin x}}.\frac{2\cos x}{(1-\sin x)^2}$$

$$=\frac{\cos x}{1-\sin x}\sqrt{\frac{1-\sin x}{1+\sin x}}.$$

$$\frac{dz}{dx}=\frac{d\ln\left(\frac{1+\cos x}{1-\cos x}\right)}{dx}$$

$$=\frac{d\ln\left(\frac{1+\cos x}{1-\cos x}\right)}{d\left(\frac{1+\cos x}{1-\cos x}\right)}.d\frac{\left(\frac{1+\cos x}{1-\cos x}\right)}{dx}$$

$$=\frac{1}{\left(\frac{1+\cos x}{1-\cos x}\right)}\times(1-\cos x)$$

$$\frac{d\frac{(1+\cos x)}{dx}-(1+\cos x)d\frac{(1-\cos x)}{dx}}{(1-\cos x)^2}$$

$$=\frac{1-\cos x}{1+\cos x}\times\frac{(1-\cos x)(-\sin x)-(1+\cos x)\sin x}{(1-\cos x)^2}$$

$$= \frac{1-\cos x}{1+\cos x}\times\frac{2\sin x}{(1-\cos x)^2}$$

$$= \frac{-2\sin x}{(1+\cos x)(1-\cos x)}$$

$$= \frac{-2\sin x}{1-\cos^2 x} = \frac{-2\sin x}{\sin^2 x} = -\frac{2}{\sin x}$$

From (1), we get

$$\frac{dy}{dz} = \frac{1}{2}.\frac{\cos x}{1-\sin x}\sqrt{\frac{1-\sin x}{1+\sin x}}\Big/-\frac{2}{\sin x}$$

$$= -\frac{\sin x\cos x}{4(1-\sin x)}\sqrt{\frac{1-\sin x}{1+\cos x}}$$

7. Given that $y = \frac{\sin^1 x}{\sqrt{1-x^2}}$ .....(1)

$\Rightarrow \quad y\sqrt{1-x^2} = \sin^{-1} x$

Squaring both sides, we get

$$y^2(1-x^2) = (\sin^{-1} x)^2$$

Differentiating both sides, we get

$$\frac{d}{dx}y^2(1-x^2) = \frac{d(\sin^{-1} x)^2}{dx}$$

$$\Rightarrow \quad 2yy_1(1-x^2) + y^2(-2x) = 2\sin^{-1} x.\frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow \quad 2yy_1(1-x^2) - 2y^2x = 2y$$

$$\Rightarrow \quad y_1(1-x^2) - yx = 2.$$

Differentiating both sides, we get

$$\frac{dy}{dx}y^2(1-x^2) - \frac{d(yx)}{dx} = \frac{d2}{dx}$$

$$\Rightarrow \quad y_2(1-x^2) + b_1(-2x) - (y_1x + y) = 0$$

$$\Rightarrow \quad y_2(1-x^2) - 3y_1x - y = 0$$

$$\Rightarrow \quad (1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$$

8. Given that $y = x^{\sin x-\cos x} + \frac{x^2-1}{x^2+1}$

Let $u = x^{\sin x-\cos x}$, $v = \frac{x^2-1}{x^2+1}$

$\therefore \quad y = u + v$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad ......(1)$$

$u = x^{\sin x-\cos x}$

$\therefore \quad \ln u = \ln\left(x^{\sin x-\cos x}\right) = (\sin x-\cos x)\ln x$

Differentiating both sides w.r.t x, we get

$$\frac{d\ln u}{dx} = \frac{d(\sin x-\cos x).\ln x}{dx}$$

$$\Rightarrow \quad \frac{1}{u}.\frac{dy}{dx} = (\sin x-\cos x)\frac{d\ln x}{dx} + \ln x.\frac{d(\sin x-\cos x)}{dx}$$

$$= \frac{1}{x}(\sin x-\cos x) + (\cos x+\sin x)\ln x$$

$$\Rightarrow \quad \frac{dy}{dx} = u.\left[\frac{1}{x}(\sin x-\cos x) + (\cos x+\sin x)\ln x\right]$$

$$= x^{\sin x-\cos x}\left[\frac{1}{x}(\sin x-\cos x) + (\cos x+\sin x)\ln x\right]$$

$$v = \frac{x^2-1}{x^2+1}$$

$$\frac{dv}{dx} = \frac{d\left(\frac{x^2-1}{x^2+1}\right)}{dx}$$

$$= \frac{(x^2+1)\frac{d(x^2-1)}{dx} - (x^2-1).d\frac{(x^2+1)}{dx}}{(x^2+1)^2}$$

$$= \frac{(x^2+1)2x - (x^2-1).2x}{(x^2+1)^2}$$

$$= 2x\frac{(x^2+1-x^2+1)}{(x^2+1)^2} = \frac{4x}{(x^2+1)^2}$$

From (1) we get

$$\frac{dy}{dx}=x^{\sin x-\cos x}\left[\frac{1}{x}(\sin x-\cos x)+(\cos x+\sin x)\ln x\right]$$

$$+\frac{4x}{(x^2+1)^2}$$

9. Equations of two curves are

$y = 2^x$ ....(1)

$y = 5^x$ ....(2)

Their point of intersection is (0, 1).

Slope of the tangent to the curve (1) is

$$\frac{dy}{dx}=2^x\ln 2$$

At the point (0, 1), $\frac{dy}{dx}=2^0.\ln 2=\ln 2$

$\Rightarrow$ $m_1 = \ln_2$

Slope of the tangent to the curve (2) is

$$\frac{dy}{dx}=5^x.\ln 5$$

At the point (0, 1), $\frac{dy}{dx}$ = 5^0. ln 5 = ln 5

$\Rightarrow$ m_2 = ln 5

Let θ be the angle between the two curves

$$\therefore\ \tan\theta=\frac{m_2-m_1}{1+m_1m_2}=\frac{\ln 5-\ln 2}{1+\ln 5.\ln 2}=\frac{\ln\left(5/2\right)}{1+\ln 2\ln 5}$$

$$\Rightarrow\ \tan^{-1}\left[\frac{\ln(5/2)}{1+\ln 2\ln 5}\right]$$

10. Let y = $x^{\frac{1}{x}}$

$$\Rightarrow\ \ln y=\ln\left(x^{\frac{1}{x}}\right)=\frac{\ln x}{x}$$

$$\frac{d\ln y}{dx}=d\frac{\left(\frac{\ln x}{x}\right)}{dx}$$

$$\Rightarrow\ \frac{1}{y}.\frac{dy}{dx}=\frac{x d\frac{\ln x}{dx}-\ln x.\frac{dx}{dx}}{x^2}$$

$$=\frac{x-\frac{1}{x}-\ln x}{x^2}=\frac{1-\ln x}{x^2}$$

$$\Rightarrow\ \frac{dy}{dx}=\frac{y(1-\ln x)}{x^2}$$

$$\frac{d^2y}{dx^2}=x^2,\ \frac{d\frac{y(1-\ln x)}{dx}-y(1-\ln x).\frac{dx^2}{dx}}{x^4}$$

$$=x^2\ \frac{\left[\frac{dy}{dx}(1-\ln x).+y\left(-\frac{1}{x}\right)\right]-2xy(1-\ln x)}{x^4}$$

$$=x^2\ \frac{\left[\frac{y(1-\ln x)^2}{x^2}-\frac{y}{x}\right]-2xy(1-\ln x)}{x^4}$$

$$=\frac{y(1-\ln x)^2-xy-2xy(1-\ln x)}{x^4}$$

For maximum or minimum, $\frac{dy}{dx}$ = 0

$$\Rightarrow\ \frac{y(1-\ln x)}{x^2}=0$$

$\Rightarrow$ 1 – ln x = 0

$\Rightarrow$ ln x = 1 = ln e

$\Rightarrow$ x = e

When x = e, $\frac{d^2y}{dx^2}$ is –ve.

So the function is maximum at x = e

maximum value of the function = $e^{\frac{1}{e}}$.

$$e^{\frac{1}{e}}>x^{\frac{1}{x}}$$

$$\Rightarrow\ e^{\frac{1}{e}}>\pi^{\frac{1}{\pi}}$$

$$\Rightarrow\ \left(e^{\frac{1}{e}}\right)^{\pi e}>\left(\pi^{\frac{1}{\pi}}\right)^{\pi e}$$

$$\Rightarrow\ e^{\pi}>\pi^{e}$$

11. The equation of the curcve is

$$\sqrt{x}+\sqrt{y}=\sqrt{a} \quad ....(1)$$

Let P be a point on the curve (1) whose coordinates are (x_1, y_1).

$$\therefore \quad \sqrt{x_1}+\sqrt{y_1}=\sqrt{a} \quad .....(2)$$

Differentiating both sides of (1), we get

$$\frac{d\sqrt{x}}{dx}+\frac{d\sqrt{y}}{dx}=\frac{d\sqrt{a}}{dx}$$

$$\Rightarrow \frac{1}{2\sqrt{x}}+\frac{1}{2\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{\sqrt{x}}+\frac{1}{\sqrt{y}}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{\sqrt{y}}\frac{dy}{dx}=-\frac{1}{\sqrt{x}}$$

$$\Rightarrow \frac{dy}{dx}=-\frac{\sqrt{y}}{\sqrt{x}}$$

At the point (x_1, y_1), $\frac{dy}{dx}=-\frac{\sqrt{y_1}}{\sqrt{x_1}}$.

Equation of the tangent at the point (x_1, y_1) is

$$y-y_1=\frac{dy}{dx}(x-x_1)$$

$$\Rightarrow y-y_1=\frac{\sqrt{y_1}}{\sqrt{x_1}}(x-x_1)$$

$$\Rightarrow x\sqrt{y_1}+y\sqrt{x_1}=\sqrt{x_1}\sqrt{y_1}\left(\sqrt{x_1}+\sqrt{y_1}\right)$$

$$\Rightarrow \sqrt{x_1}\sqrt{y_1}\sqrt{a}$$

$$\Rightarrow \frac{x}{\sqrt{x_1}\sqrt{a}}+\frac{y}{\sqrt{y_1}\sqrt{a}}=1 \quad .....(3)$$

Sum of the x-intercepts & y-intercepts

$$=\sqrt{x_1}\sqrt{a}+\sqrt{y_1}\sqrt{a}$$

$$=\sqrt{a}\left(\sqrt{x_1}+\sqrt{y_1}\right)$$

$$=\sqrt{a}.\sqrt{a}=a$$ which is a constant.

12. Let R be the radius of the sphere.

Let ABCD be the inscribed cylinder.

Let 2x be the radius of the base and 2y be the height of the cylinder.

Here $R^2 = x^2 + y^2$

$\Rightarrow x^2 = R^2 - y^2$ where R is a constant.

Let V be the volume of the cylinder.

$$V=\pi x^2 y$$

$$=\pi\left(R^2-y^2\right)y$$

$$=\pi\left(R^2y-y^3\right)$$

$$\frac{dv}{dy}=\pi\left(R^2-3y^2\right)$$

$$\frac{d^2v}{dy^2}=\pi(-6y)=-6\pi y.$$

For maximum or minimum, $\frac{dv}{dy}=0$

$$\Rightarrow \pi\left(R^2-3y^2\right)=0$$

$$\Rightarrow R^2-3y^2=0$$

$$\Rightarrow 3y^2 = R^2$$

$$\Rightarrow y^2=\frac{R^2}{3}\Rightarrow y=\frac{R}{\sqrt{3}}$$

When $y=\frac{R}{\sqrt{3}}$, $\frac{d^2v}{dy^2}=-6\pi.\frac{R}{\sqrt{3}}$ which is –ve.

For maximum volume, altitude of the cylinder

$$=2y=\frac{2R}{\sqrt{3}}.$$

13. Let θ be the semi vertical angle of the cone.

Here $\theta \in \left(0, \frac{\pi}{2}\right)$

Let r, h and l be the radius of the base, height and the slant height of the cone ℓ is constant.

From ΔABC, r = $\ell \sin\theta$, $h = \ell\cos\theta$.

Let V be the volume of the cone.

$$V = \frac{1}{3}\pi r^2 h.$$

$$= \frac{1}{3}\pi \ell^2 \sin^2\theta . \ell\cos\theta$$

$$= \frac{1}{3}\pi \ell^3 \sin^2\theta\cos\theta$$

Differentiating w.r.t θ, we get

$$\frac{dv}{d\theta} = \frac{1}{3}\pi\ell^3\left[\sin^2\theta.\frac{d\cos\theta}{d\theta} + \cos\theta.\frac{d\sin^2\theta}{d\theta}\right]$$

$$= \frac{1}{3}\pi\ell^3\left[\sin^2\theta - \sin\theta + \cos\theta.2\sin\theta\cos\theta\right]$$

$$= \frac{1}{3}\pi\ell^3\left[-\sin^3\theta + 2\sin\theta\cos^2\theta\right]$$

$$\frac{d^2v}{d\theta^2} = \frac{1}{3}\pi\ell^3 - [3\sin^2\theta\cos\theta + 2\sin\theta.2\cos\theta$$

$$(-\sin\theta) + 2\cos^2\theta.\cos\theta]$$

$$= \frac{1}{3}\pi\ell^3\left(2\cos^3\theta - 7\sin^2\theta\cos\theta\right)$$

For maximum or minimum, $\frac{dv}{d\theta} = 0$

$\Rightarrow -\sin^3\theta + 2\sin\theta\cos^2\theta = 0$

$\Rightarrow \sin^3\theta = 2\sin\theta\cos^2\theta$

$\Rightarrow \tan^2\theta = 2$ ......(1)

$\Rightarrow \tan\theta = \sqrt{2}$

$\Rightarrow \theta = \tan^{-1}\sqrt{2}$

From (1), we get $\frac{\sin^2\theta}{\cos^2\theta} = 2$

$\Rightarrow \sin^2\theta = 2\cos^2\theta$

$$\frac{d^2v}{d\theta^2} = \frac{1}{3}\pi\ell^3 \times -12\cos^3\theta$$

= $-4\pi e^3\cos^3\theta$ which is –ve.

$\therefore$ V is maximum when $\theta = \tan^{-1}\sqrt{2}$

$\Rightarrow \tan\theta = \sqrt{2}$

$\Rightarrow \cos\theta = \frac{1}{\sqrt{2+1}} = \frac{1}{\sqrt{3}}$

$\Rightarrow \theta = \cos^{-1}\frac{1}{\sqrt{3}}$

14. The equation of the curve is

$y^2 - x^2 + 2x - 1 = 0$ .....(1)

Differentiating both sides of (1), w.r.t, we get

$$\frac{dy^2}{dx} - \frac{dx^2}{dx} + \frac{d2x}{dx} - \frac{d1}{dx} = 0$$

$\Rightarrow 2y\frac{dy}{dx} - 2x + 2 = 0$

$\Rightarrow y\frac{dy}{dx} = x - 1$

$\Rightarrow \frac{dy}{dx} = \frac{x-1}{y}$.

Since the tangent is parallel to x-axis,

$\frac{dy}{dx} = 0$

$\Rightarrow \frac{x-1}{y} = 0$

$\Rightarrow$ x = 1.

When x = 1, then from (1), we have

$y^2 - 1 + 2 - 1 = 0$

$\Rightarrow y^2 = 0 \Rightarrow y = 0$

The required point is (1, 0).

15. Equation of the curve is

$$\frac{a^2}{x^2}+\frac{b^2}{y^2}=1 \qquad .....(1)$$

Let P be any point on the curve whose coordinates are (a sec θ, b cosec θ).

The distance of P from the origin is

$$s=\sqrt{a^2\ \sec^2\ \theta + b^2\ \mathrm{cosec}^2\theta}$$

$$\frac{ds}{d\theta}=\frac{d\left(a^2\sec^2\theta+b^2\mathrm{cosec}^2\theta\right)^{\frac{1}{2}}}{d\theta}$$

$$= \frac{1}{2}\left(a\sec^2\theta+b^2\mathrm{cosec}^2\theta\right)^{-\frac{1}{2}}$$

$$\times\left(2a^2\sec^2\theta\tan\theta-2b^2\mathrm{cosec}^2\theta\cot\theta\right)$$

$$= \frac{a^2\sec^2\theta\tan\theta-b^2\mathrm{cosec}^2\theta\cot\theta}{\sqrt{a^2\sec^2\theta+b^2\mathrm{cosec}^2\theta}}$$

For maximum or minimum, $\frac{ds}{d\theta}=0$

$$\Rightarrow\quad a^2\sec^2\theta\tan\theta-b^2\mathrm{cosec}^2\theta\cot\theta=0$$

$$\Rightarrow\quad a^2\frac{\sin\theta}{\cos^3\theta}=b^2\frac{\cos\theta}{\sin^3\theta}$$

$$\Rightarrow\quad \tan^4\theta=\frac{b^2}{a^2}\Rightarrow\tan^2\theta=\frac{b}{a}\Rightarrow\tan\theta=\frac{\sqrt{y}}{\sqrt{a}}$$

$$\Rightarrow\quad \frac{\sin\theta}{\cos\theta}=\frac{\sqrt{b}}{\sqrt{a}}$$

$$\Rightarrow\quad \frac{\sin\theta}{\sqrt{b}}=\frac{\cos\theta}{\sqrt{a}}=\frac{\sqrt{\sin^2\theta+\cos^2}}{\sqrt{a+b}}=\frac{1}{\sqrt{a+b}}$$

$$\Rightarrow\quad \sin\theta=\frac{\sqrt{b}}{\sqrt{a+b}},\ \cos\theta=\frac{\sqrt{a}}{\sqrt{a+b}}$$

For the above values of $\sin\theta$, $\cos\theta$, $\frac{d^2s}{d\theta^2}>0$

So S is minimum when

$$\sin\theta=\frac{\sqrt{b}}{\sqrt{a+b}},\cos\theta=\frac{\sqrt{a}}{\sqrt{a+b}}$$

$$S\ =\ \sqrt{a^2\sec^2\theta+b^2\mathrm{cosec}^2\theta}$$

$$=\ \sqrt{a^2\left(\frac{a+b}{a}\right)+b^2\left(\frac{a+b}{b}\right)}$$

$$=\ \sqrt{a(a+b)+b(a+b)}$$

$$=\ \sqrt{(a+b)^2}=a+b.$$

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10. Integrals, Application of Integrals, Differential Equations. Each question carries 6 marks.

1. Let $I = \int \frac{1}{\cos x(1+2\sin x)} dx$

$= \int \frac{\cos x\ dx}{\cos^2 x(1+2\sin x)} dx$

$= \int \frac{\cos x\ dx}{(1-\sin^2 x)(1+2\sin x)}$

$= \int \frac{\cos x}{(1-\sin x)(1+\sin x)(1+2\sin x)} dx$

Let sin x = t

$\therefore \quad \cos x\, dx = dt$

$I = \int \frac{dt}{(1-t)(1+t)(1+2t)}$

Let $\frac{1}{(1-t)(1+t)(1+2t)} = \frac{A}{1-t} + \frac{B}{1+t} + \frac{c}{1+2t}$

$\Rightarrow$ 1 = A (1+t) (1+2t) + B (1-t)(1+2t)+c(1-t)(1+t)

Putting t = 1, we get $A = \frac{1}{6}$

Putting t = -1, we get $B = -\frac{1}{2}$

Putting $t = -\frac{1}{2}$, we get $c = \frac{4}{3}$

$\therefore \quad \frac{1}{(1-t)(1+t)(1+2t)}$

$= \frac{1}{6}.\frac{1}{1-t} - \frac{1}{2}.\frac{1}{1+t} + \frac{4}{3}.\frac{1}{1+2t}$

Integrating both sides we get

$I = \int \frac{1}{(1-t)(1+t)(1+2t)} dt$

$= \frac{1}{6}\int \frac{1}{1-t} dt - \frac{1}{2}\int \frac{1}{1+t} dt + \frac{4}{3}\int \frac{1}{1+2t} dt$

$= -\frac{1}{6}\ln(1-t) - \frac{1}{2}\ln(1+t) + \frac{2}{3}.\ln(1+2t+c)$

$= -\frac{1}{6}\ln(1-\sin x) - \frac{1}{2}\ln(1+\sin x)$

$+ \frac{2}{3}\ln(1+2\sin x) + c$

2. Let $I = \int_0^{\pi} \frac{x}{1+\sin x} dx$

$= \int_0^{\pi/2} \frac{x}{1+\sin x} dx + \int_0^{\pi/2} \frac{\pi - x}{1+\sin(\pi - x)} dx$

$= \int_0^{\pi/2} \frac{x}{1+\sin x} dx + \int_0^{\pi/2} \frac{\pi - x}{1+\sin x} dx$

$= \int_0^{\pi/2} \frac{x}{1+\sin x} dx + \pi\int_0^{\pi/2} \frac{1}{1+\sin x} dx$

$- \int_0^{\pi/2} \frac{x}{1+\sin x} dx$

$= \pi\int_0^{\pi/2} \frac{1}{1+\sin x} dx$

$= \pi\int_0^{\pi/2} \frac{1}{1+\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}} dx$

$= \pi\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 1} dx$

$= \pi\int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{\left(\tan\frac{x}{2}+1\right)^2} dx$

Let $\tan\frac{x}{2} + 1 = t$

$\sec^2\frac{x}{2} dx = 2dt$

When x = 0, t = 1

When $x = \frac{\pi}{2}$ t = 2

$I = \pi\int_1^2 \frac{2dt}{t^2} = 2\pi\int_1^2 t^{-2} dt$

$$= 2\pi\left[\frac{t^{-1}}{-1}\right]_1^2 = 2\pi\left[-\frac{1}{t}\right]_1^2$$

$$= 2\pi\left[-\frac{1}{2}-\left(-\frac{1}{1}\right)\right] = 2\pi\left[-\frac{1}{2}+1\right]$$

$$= 2\pi - \frac{1}{2} = \pi$$

3. Let $I = \int_{\pi/6}^{\pi/3} \frac{1}{1+\sqrt[4]{\cos x}}dx$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt[4]{\sin x}}{\sqrt[4]{\sin x}+\sqrt[4]{\cos x}}dx \quad ....(1)$$

or $I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[4]{\sin\left(\frac{\pi}{3}+\frac{\pi}{5}-x\right)}}{\sqrt[4]{\sin\left(\frac{\pi}{3}+\frac{\pi}{5}-x\right)}+\sqrt[4]{\cos\left(\frac{\pi}{3}+\frac{\pi}{5}-x\right)}}dx$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt[4]{\sin\left(\frac{\pi}{2}-x\right)}}{\sqrt[4]{\sin\left(\frac{\pi}{2}-x\right)}+\sqrt[4]{\cos\left(\frac{\pi}{2}-x\right)}}dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt[4]{\cos x}}{\sqrt[4]{\cos x}+\sqrt[4]{\sin x}}dx \quad ....(2)$$

Adding (1) and (2), we get

$$2I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[4]{\sin x}}{\sqrt[4]{\sin x}+\sqrt[4]{\cos x}}dx + \int_{\pi/6}^{\pi/3} \frac{\sqrt[4]{\cos x}}{\sqrt[4]{\cos x}+\sqrt[4]{\sin x}}dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\sqrt[4]{\sin x}\sqrt[4]{\cos x}}{\sqrt[4]{\sin x}+\sqrt[4]{\cos x}}dx$$

$$= \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3}$$

$$= \frac{\pi}{3}-\frac{\pi}{6}=\frac{\pi}{6}$$

$$\Rightarrow I = \frac{\pi}{12}$$

4. Let $I = \int_0^1 \frac{\log(1+x)}{1+x^2}dx$

Let $x = \tan\theta$

$\Rightarrow dx = \sec^2\theta\, d\theta$

when x = 0, $\theta = 0$

when x = 1, $\theta = \frac{\pi}{4}$

$$\therefore I = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{1+\tan^2\theta}.\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1+\tan\theta)d\theta \quad ....(1)$$

$$= \int_0^{\pi/4} \log\left[1+\tan\left(\frac{\pi}{4}-\theta\right)\right]d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}.\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \log\left[1+\frac{1-\tan\theta}{1+\tan\theta}\right]d\theta$$

$$= \int_0^{\pi/4} \log\left(\frac{2}{1+\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/4} \log 2\, d\theta - \int_0^{\pi/4} \log(1+\tan\theta)d\theta$$

$$= \log 2[\theta]_0^{\pi/4} - I$$

$$\Rightarrow 2I = \log 2.\frac{\pi}{4} = \frac{\pi \log 2}{4}$$

$$\Rightarrow I = \frac{\pi}{8}\log 2.$$

5. Let $I = \int \frac{1}{2\cos^2 x+\cos x}dx$

$$= \int \frac{1}{\cos x(2\cos x+1)}dx$$

We know $\frac{1}{\cos x(2\cos x+1)}$

$= \frac{1}{\cos x}-\frac{2}{2\cos x+1}$

$$I=\int\frac{1}{\cos x}dx-\int\frac{1}{2\cos x+1}dx$$

$$=\int\sec x\,dx-\int\frac{1}{2.\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+1}dx$$

$$=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)-\int\frac{1+\tan^2\frac{x}{2}}{2-2\tan^2\frac{x}{2}+1+\tan^2\frac{x}{2}}$$

$$=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)-\int\frac{\sec^2\frac{x}{2}}{3-\tan^2\frac{x}{2}}dx$$

Let = ln tan $\left(\frac{\pi}{2}=t\right)$

$$\sec^2\frac{x}{2}dx=2dt$$

$$I=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)-\int\frac{2dt}{3-t^2}$$

$$=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+2\int\frac{dt}{t^2-\left(\sqrt{3}\right)^2}$$

$$=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+2.\frac{1}{2\sqrt{3}}\ln\frac{t-\sqrt{3}}{t+\sqrt{3}}+C$$

$$=\ln\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)+\frac{1}{\sqrt{3}}\ln\left(\frac{\tan\frac{x}{2}-\sqrt{3}}{\tan\frac{x}{2}+\sqrt{3}}\right)+C$$

6. Let $I=\int_0^{\pi/2}\frac{\sin x(7-\cos x)}{(1+\cos^2 x)(2-\cos x)}dx$

Let cos x = t

$\Rightarrow$ – sin x dx = dt

$\Rightarrow$ sin x dx = – dt

When x = 0, t = cos 0 = 1

When $x=\frac{\pi}{2}$ $t=\cos\frac{\pi}{2}=0$

$$I=\int_1^0\frac{(7-t).-dt}{(1+t^2)(2-t)}$$

$$=\int_0^1\frac{(7-t)}{(1+t^2)(2-t)}dt \quad ......(1)$$

Let $\frac{7-t}{(1+t^2)(2-t)}=\frac{At+B}{1+t^2}+\frac{C}{2-t}$ ....(2)

$\therefore$ 7–t = (At+B)(2–t)+C(1+t^2)

When t = 2, we get

7–2 = (2A+B).O + C(1+4)

$\Rightarrow$ C = 1

Equating the coefficient of t^2, we get

O = –A + C

$\Rightarrow$ O = –A+1 $\Rightarrow$ A = 1

Equating the coefficient of

2A – B = –1

$\Rightarrow$ 2 – B = –1 $\Rightarrow$ B = 3

From (2), we get

$$\frac{7-t}{(1+t^2)(2-t)}=\frac{t+3}{1+t^2}+\frac{1}{2-t}$$

$$I=\int_0^1\frac{7-t}{(1+t^2)(2-t)}dt=\int_0^1\frac{t+3}{1+t^2}+\int_0^1\frac{1}{2-t}dt$$

$$=\frac{1}{2}\int_0^1\frac{2t}{1+t^2}dt+3\int_0^1\frac{1}{1+t^2}dt-\left[\ln(2-t)\right]_0^1$$

$$=\frac{1}{2}\ln 2+3\tan^{-1}1+\ln 2$$

$$=\frac{3}{2}\ln 2+3\frac{\pi}{4}$$

7. The equation of the parabola is

$y^2 = x$ ....(1)

The equation of the circle is

$x^2 + y^2 = 2x$ ....(2)

Solving (1) and (2), we get

$x^2 + x = 2x$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x(x-1) = 0$

$\Rightarrow x = 0, 1$.

The points of intersection of the parabola (1) and the circle (2) are (0, 0), (1,1) and (1, -1).

Required area

$= 2 \times \text{Area of OAB}$

$= 2\left[\int_0^1 \sqrt{2x - x^2}\,dx - \int_0^1 \sqrt{x}\,dx\right]$

$= 2\left[\int_0^1 \sqrt{1-(1-x)^2}\,dx - \frac{2}{3}\right]$

$= 2\left[\frac{(1-x)\sqrt{1-(1-x)^2}}{2} + \frac{1}{2}\sin^{-1}\left(\frac{1-x}{1}\right)\right]_0^1 - \frac{4}{3}$

$= \frac{\pi}{2} + \frac{4}{3}$ units is magnitude.

8. The equation of the two curves are

$y^2 = 2x$ ......(1)

$y = 4x - 1$ ....(2)

The intersection of (1) and (2) are

$\left(\frac{1}{8}, -\frac{1}{2}\right)$ and $\left(\frac{1}{2}, 1\right)$.

Required area

$= \int_0^{1/2} \sqrt{2x}\,dx - 4\int_{1/8}^{1/2} (4x-1)\,dx$

$= \sqrt{2}\int_{1/8}^{1/2} x^{\frac{1}{2}}\,dx - 4\int_{1/8}^{1/2} x\,dx + \int_{1/8}^{1/2} dx$

$= \sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_{\frac{1}{8}}^{\frac{1}{2}} - 4\left[\frac{x^2}{2}\right]_{1/8}^{1/2} + [x]_{1/8}^{1/2}$

$= \sqrt{2}.\frac{2}{3}\left[x\sqrt{x}\right]_{1/8}^{1/2} - 2\left[x^2\right]_{1/8}^{1/2} + \frac{1}{2} - \frac{1}{8}$

$= \frac{2\sqrt{2}}{3}\left[\frac{1}{2\sqrt{2}} - \frac{1}{8.2\sqrt{2}}\right] - 2\left[\frac{1}{4} - \frac{1}{64}\right] + \frac{3}{8}$

$= \frac{2\sqrt{2}}{3}\frac{7}{16\sqrt{2}} - \frac{15}{32} + \frac{3}{8}$

$= \frac{7}{24} - \frac{15}{32} + \frac{3}{8} = \frac{1}{24}$ sq. unit in magnitude.

9. The equation of the ellipse is

$\frac{x^2}{12} + \frac{y^2}{16} = 1$

$\Rightarrow \frac{y^2}{16} = 1 - \frac{x^2}{12} = \frac{12 - x^2}{12}$

$\Rightarrow y^2 = \frac{16}{12}\left(12 - x^2\right)$

$\Rightarrow y = \frac{4}{2\sqrt{3}}\sqrt{12 - x^2}$

Required area $= \int_{-2\sqrt{3}}^{3} y\,dx$

$= \int_{-2\sqrt{3}}^{3} \frac{4}{2\sqrt{2}}\sqrt{12 - x^2}\,dx$

$= \frac{4}{2\sqrt{3}}\int_{-2\sqrt{3}}^{3} \sqrt{\left(2\sqrt{3}\right)^2 - x^2}\,dx$

$= \frac{4}{2\sqrt{3}}\left[x\frac{\sqrt{\left(2\sqrt{3}\right) - x^2}}{2} + \frac{\left(2\sqrt{3}\right)^2}{2}\sin^{-1}\frac{x}{2\sqrt{3}}\right]^3$

$= \frac{4}{2\sqrt{3}}\left[\frac{3\sqrt{3}}{2} + 6\sin^{-1}\frac{\sqrt{3}}{2} - \left\{0 + 6\sin^{-1}(-1)\right\}\right]$

$= \frac{4}{2\sqrt{3}}\left[\frac{3\sqrt{3}}{2} + 6.\frac{\pi}{3} - 6\left(-\frac{\pi}{2}\right)\right]$

$= \frac{2}{\sqrt{3}}\left[\frac{3\sqrt{3}}{2} + 5\pi\right]$ sq. unit.

10. The equation of the parabola is $y^2 = 4x$ ....(1)
The equation of the line is $y = 2x$. ......(2)

From (1) & (2), we get

$$4x^2 = 4x$$

$$\Rightarrow 4x(x-1) = 0$$

$$\Rightarrow x = 0, 1$$

When $x = 0$, $y = 0$

When $x = 1$, $y = 2$

The points of intersection of the parabola and the line are (0, 0) and (1, 2).

The area enclosed by the parabola & the line

$$= \int_0^1 \left(\sqrt{4x} - 2x\right) dx$$

$$= 2\int_0^1 \sqrt{x}\, dx - 2\int_0^1 x\, dx$$

$$= 2\left[\frac{x^{3/2}}{3/2}\right]_0^1 - 2\left[\frac{x^2}{2}\right]_0^1$$

$$= 2.\frac{2}{3}.1 - 1$$

$$= \frac{4}{3} - 1 = \frac{1}{8}. \text{ sq. unit.}$$

11. Given differential equation is

$$\frac{dx}{dy} = \frac{3x - 7y + 7}{3y - 7x - 3}$$

$$\Rightarrow \frac{dy}{dx} - \frac{3y - 7x - 3}{3x - 7y - 3} \quad ...(1)$$

Let $x = x + h, \quad y = Y + K$

$$\therefore \quad \frac{dy}{dx} = \frac{dY}{dX}.$$

Let us choose hand k such that the reduced equation (1) will be homogeneous.

From (10 we have

$$\frac{dY}{dX} = \frac{3(Y+k) - 7(X+4) - 3}{3(X+4) - 7(Y+k) + 7}$$

$$\Rightarrow \frac{dY}{dx} = \frac{(3Y - 7x) - 74 + 3h - 3}{(3x - 7Y) - 3h - 7k + 7} \quad ....(2)$$

Let us choose h and k such that

$$-7h + 3k - 3 = 0$$

$$3h - 7k - 7 = 0$$

Solving the above equations, we get

$$h = -\frac{21}{20}, \; k = -\frac{29}{20}$$

$$\therefore \quad x = X - \frac{21}{20} \text{ and } y = Y - \frac{29}{20}.$$

$$\Rightarrow \quad X = x + \frac{21}{20}, \quad Y = y + \frac{29}{20}.$$

Equation (2) becomes

$$\frac{dY}{dx} = \frac{3Y - 7x}{3x - 7Y} \quad ......(3)$$

Let Y = VX

$$\Rightarrow \frac{dY}{dx} = V + X\frac{dV}{dX}$$

Equation (3) becomes

$$\frac{dY}{dX} = \frac{3VX - 7X}{3X - 7VX} = \frac{3V - 7}{-7V + 3}$$

$$\Rightarrow V + X\frac{dv}{dx} = \frac{3V - 7}{-7V + 3}$$

$$\Rightarrow X\frac{dV}{dX} = \frac{3V - 7}{-7V + 3} - V$$

$$= \frac{3V - 7 - V(-7V + 3)}{-7V + 3}$$

$$= \frac{3V - 7 + 7V^2 - 3V}{-7V + 3}$$

$$= \frac{7(V^2 - 1)}{-7V + 3}$$

$$\Rightarrow \frac{-7V + 3}{V^2 - 1} dV = 7\frac{dx}{X}$$

Integrating both sides, we get

$$\int \frac{-7V + 3}{V^2 - 1} dV = 7\int \frac{dx}{X}$$

$$= -\frac{7}{2}\int \frac{2V}{V^2 - 1} dV + 3\int \frac{dV}{V^2 - 1} = 7 \ln X + C$$

$$= -\frac{7}{2}\ln(V^2 - 1) + 3\frac{1}{2}\ln\frac{V-1}{V+1} = 7\ln X + \ln C$$

$$\Rightarrow \ln\left(V^2-1\right)^{\frac{-7}{2}}+\ln\left(\frac{v-1}{v+1}\right)^{3/2}=\ln X^7+\ln C$$

$$\Rightarrow \ln\left(V^2-1\right)^{\frac{-7}{2}}\left(\frac{V-1}{V+1}\right)^{3/2}=\ln C X^7$$

$$\Rightarrow \left(V^2-1\right)^{\frac{-7}{2}}\left(\frac{V-1}{V+1}\right)^{3/2}=C X^7$$

$$\Rightarrow \left(\frac{Y^2}{X^2}-1\right)^{\frac{-7}{2}}\left(\frac{\frac{Y}{X}-1}{\frac{Y}{X}+1}\right)^{3/2}=CX^7$$

$$\Rightarrow \frac{X^{7/2}}{(Y-x)^{7/2}}\left(\frac{Y-X}{Y+X}\right)^{3/2}=CX^7$$

Where $X = x + \frac{21}{20}$, $\quad Y=y+\frac{29}{20}$

12. The given differential equation is

$$x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)+x-y\sin\frac{y}{x}=0$$

$$\Rightarrow x\frac{dy}{dx}\sin\left(\frac{y}{x}\right)=y\sin\frac{y}{x}-x$$

$$\Rightarrow \frac{dy}{dx}\sin\left(\frac{y}{x}\right)=\frac{y}{x}\sin\frac{y}{x}-1.$$

$$\Rightarrow \frac{dy}{dx}=\frac{y}{x}-\frac{1}{\sin\frac{y}{x}} \qquad \text{......(1)}$$

Let y = vx

$$\therefore \frac{dy}{dx}=v+x\frac{dv}{dx}$$

From (1), we get

$$V+x\frac{dV}{dx}=V-\frac{1}{\sin V}$$

$$\Rightarrow x\frac{dV}{dx}=-\frac{1}{\sin V}$$

$$\Rightarrow \sin V\,dV=-\frac{dx}{x}$$

Integrating both sides, we get

$$\int \sin V\,dV=-\int\frac{dx}{x}$$

$$\Rightarrow -\cos V=-\ln x-C$$

$$\Rightarrow \cos\frac{y}{x}=\ln x+C \qquad \text{.......(2)}$$

When x = 1, $\quad y=\frac{\pi}{2}$

From (2), we get

$$\cos\frac{\pi}{2}=\ln 1+C \Rightarrow C=O$$

Required particular solution is

$$\frac{y}{x}=\ln x$$

$$\Rightarrow y = x\ln x$$

13. The given differential equation is

$$x\,dy - y\,dx = \sqrt{x^2+y^2}\,dx$$

$$\Rightarrow x\,dy=y\,dx+\sqrt{x^2+y^2}\,dx$$

$$\Rightarrow \frac{dy}{dx}=\frac{y+\sqrt{x^2+y^2}}{x} \qquad \text{.......(1)}$$

Let y = Vx

$$\therefore \frac{dy}{dx}=V+x\frac{dv}{dx}$$

From (1), we get

$$V+x\frac{dv}{dx}=\frac{Vx+\sqrt{x^2+v^2x^2}}{x}$$

$$\Rightarrow V+x\frac{dv}{dx}=V+\sqrt{1+v^2}$$

$$\Rightarrow x\frac{dv}{dx}=\sqrt{1+V^2}$$

$$\Rightarrow \frac{dv}{\sqrt{1+v^2}}=\frac{dx}{x}$$

Integrating both sides, we get

$$\int\frac{dv}{\sqrt{1+v^2}}=\int\frac{dx}{x}$$

$$\ln\left(v+\sqrt{1+v^2}\right)=\ln x+\ln C$$

$$\Rightarrow \ln\left(v+\sqrt{1+v^2}\right)=\ln Cx$$

$$\Rightarrow V+\sqrt{1+v^2}=Cx$$

$$\Rightarrow \frac{y}{x}+\sqrt{1+\frac{y^2}{x^2}}=Cx$$

$$\Rightarrow y+\sqrt{x^2+y^2}=Cx^2.$$

14. The given differential equation is

$$\frac{dy}{dx}-y\cot x=xy^y$$

$$\Rightarrow \frac{1}{y^4}\frac{dy}{dx}-\frac{1}{y^3}\cot x=x \qquad ......(1)$$

Let $\frac{1}{y^3}=V$

$$\Rightarrow -3y^{-4}\frac{dy}{dx}=\frac{dv}{dx}$$

$$\Rightarrow \frac{1}{y^4}\frac{dy}{dx}=-\frac{1}{3}\frac{dv}{dx}$$

Equation (1) becomes

$$-\frac{1}{3}\frac{dv}{dx}-v\cot x=x$$

$$\Rightarrow \frac{dv}{dx}+3\cot x.v=-3x \qquad ....(2)$$

Integrating factor = $e^{\int 3\cot x\,dx}$

$$= e^{3\ln\sin x}=e^{\ln\sin^3 x}=\sin^3 x$$

Multiplying both sides of (2) by $\sin^3x$, we get

$$\sin^3 x.\frac{dv}{dx}+3\cot x.\sin^3 x.\;\; V=3-3x\sin^3 x$$

$$\Rightarrow \frac{dy}{dx}\left(V\sin^3 x\right)=-3x\sin^3 x$$

Integrating both sides, we get

$$V\sin^3 x=-3\int x\sin^3 x\,dx$$

$$\Rightarrow \frac{1}{y^3}\sin^3 x=-3\int x.\left(\frac{3}{4}\sin x-\frac{1}{4}\sin 3x\right)dx$$

$$\Rightarrow \frac{\sin^3 x}{y^3}=-\frac{9}{4}\int x\sin x\,dx+\frac{3}{4}\int x\sin 3x\,dx$$

$$\Rightarrow \frac{\sin^3 x}{y^3}=\frac{9}{4}x\cos x-\frac{9}{4}\sin x-\frac{9}{12}x\cos 3x$$

$$+\frac{9}{12}\left(\frac{\cos 3x}{3}\right)+C$$

15. Given differential equation is

$$\frac{dy}{dx}+2y\tan x=\sin x \qquad .....(1)$$

I.F. = $e^{\int 2\tan x\,dx}=e^{2\ln\sec x}=e^{\ln\sec^2 x}=\sec^2 x$

Multiplying both sides of (1) by $\sec^2x$,we get

$$\sec^2 x.\frac{dy}{dx}+2y\tan x.\sec^2 x=\sec x.\sec^2 x$$

$$\Rightarrow \frac{d}{dx}\left(y\sec^2 x\right)=\frac{\sin x}{\cos^2 x}$$

Integrating both sides, we get

$$y\sec^2 x=\int\frac{\sin x}{\cos^2 x}dx$$

$$\Rightarrow y\sec^2 x=\int-\frac{dt}{t^2}$$

$$\Rightarrow y\sec^2 x=\frac{1}{\cos x}+C \qquad .....(2)$$

When $x=\frac{\pi}{3}$, $y=0$

From (2), we get

$$O.\sec^2\frac{\pi}{3}=\frac{1}{\cos\frac{\pi}{3}}+C$$

$\Rightarrow$ = O = 2 + C $\Rightarrow$ C = –2

Required particular solution is

y $\sec^2x$ = sec x–2

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11. Vectors, Three dimensional Geometry. Each question carries 6 marks.

1. Given that $\vec{a} = 2\hat{i} + \hat{k} = 2\hat{i} + 0\hat{j} + \hat{k}$

$\vec{b} = \hat{i} + \hat{j} + \hat{k}$

$\vec{c} = 4\hat{i} - 3\hat{j} + 7\hat{k}$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

Given that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 4 & -3 & 7 \\ 1 & 1 & 1 \end{vmatrix}$$

$$\Rightarrow \hat{i}(y-z) - \hat{j}(x-2) + \hat{k}(x-y)$$

$$= \hat{i}(-3-7) - \hat{j}(4-7) + \hat{k}(4+3)$$

$$\Rightarrow \hat{i}(y-z) + \hat{j}(z-x) + \hat{k}(x-y)$$

$$= -10\hat{i} + 3\hat{j} + 7\hat{k}$$

$\therefore$ $y - z = -10$ .....(1)

$z - x = 3$ .....(2)

$x - y = 7$ .....(3)

Again given that $\vec{r}.\vec{a} = 0$

$$\Rightarrow \left(x\hat{i} + y\hat{j} + z\hat{k}\right).\left(2\hat{i} + 0\hat{j} + \hat{k}\right) = 0$$

$\Rightarrow$ $2x + z = 0$ .....(4)

Solving (1), (2), (3) & (4), we get

$x = -1$, $y = -8$, $2 = 2$

$\therefore$ $\vec{r} = -\hat{i} - 8\hat{j} + 2\hat{k}$

2. Given that

$$\vec{a} \times \left(\vec{b} \times \vec{c}\right) + \vec{b} \times \left(\vec{c} \times \vec{a}\right) + \vec{c} \times \left(\vec{a} \times \vec{b}\right) = \vec{0}$$

$$= (\vec{a}.\vec{c})\vec{b} - (\vec{a}.\vec{b})\vec{c} + (\vec{b}.\vec{a})\vec{c} - (\vec{b}.\vec{c})\vec{a} + (\vec{c}.\vec{b})\vec{a} - (\vec{c}.\vec{a})\vec{b} = \vec{0}$$

So the vectors $\vec{a} \times \left(\vec{b} \times \vec{c}\right), \vec{b} \times \left(\vec{c} \times \vec{a}\right)$, $\vec{c} \times \left(\vec{a} \times \vec{b}\right)$ are coplanar.

3. Given that $|\vec{a}| = 5$, $|\vec{b}| = 12$, $|\vec{c}| = 13$

Also given that $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$$\Rightarrow \left(\vec{a} + \vec{b} + \vec{c}\right).\left(\vec{a} + \vec{b} + \vec{c}\right) = \vec{0}.\left(\vec{a} + \vec{b} + \vec{c}\right)$$

$$\Rightarrow \vec{a}.\vec{a} + \vec{a}.\vec{b} + \vec{a}.\vec{c} + \vec{b}.\vec{a} + \vec{b}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} + \vec{c}.\vec{b} + \vec{c}.\vec{c} = 0$$

$$\Rightarrow \vec{a}^2 + \vec{b}^2 + \vec{c}^2 + 2\vec{a}.\vec{b} + 2\vec{b}.\vec{c} + 2\vec{c}.\vec{a} = 0$$

$$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + |\vec{c}|^2 + 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = 0$$

$$\Rightarrow 25 + 144 + 169 + 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = 0$$

$$\Rightarrow 2\left(\vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a}\right) = -338$$

$$\Rightarrow \vec{a}.\vec{b} + \vec{b}.\vec{c} + \vec{c}.\vec{a} = -169$$

4. ABC is any triangle

$\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$, $\overrightarrow{AB} = \vec{c}$

We know $\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{0}$

$$\Rightarrow \vec{a} + \vec{b} + \vec{c} = \vec{0}$$

$$\Rightarrow \vec{a} = -\vec{b} - \vec{c}$$

$$\Rightarrow \vec{a}.\vec{a} = \left(-\vec{b} - \vec{c}\right).\vec{a}$$

$$\Rightarrow |\vec{a}|^2 = -\vec{b}.\vec{a} - \vec{c}.\vec{a}$$

$$= |\vec{a}|\vec{b}\cos(180 - C) - |\vec{c}||\vec{a}|\cos(180 - B)$$

$$\Rightarrow a^2 = -ab(-\cos c) - ca(-\cos B)$$

$$\Rightarrow a^2 = ab\cos C + ca\cos B$$

$= a\,(b \cos C + c \cos B)$

$\Rightarrow$ $a = b \cos C + c \cos B$

5. ABC is any triangle.

Let $\overrightarrow{BC} = \vec{a}$, $\overrightarrow{CA} = \vec{b}$

and $\overrightarrow{AB} = \vec{c}$.

$$\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB} = \vec{O}$$

$$\Rightarrow \quad \vec{a} + \vec{b} + \vec{c} = \vec{O}$$

$$\Rightarrow \quad \vec{a} = -\left(\vec{b} + \vec{c}\right)$$

$$\Rightarrow \quad \vec{a}.\vec{a} = -\left(\vec{b} + \vec{c}\right).-\left(\vec{b} + \vec{c}\right)$$

$$\Rightarrow \quad |\vec{a}|^2 = \left(\vec{b} + \vec{c}\right).\left(\vec{b} + \vec{c}\right)$$

$$= \vec{b}^2 + \vec{c}^2 + 2\vec{b}.\vec{c}$$

$$= |\vec{b}|^2 + |\vec{c}|^2 + 2|\vec{b}||\vec{c}|\cos\left(180^0 - A\right)$$

$\Rightarrow$ $a^2 = b^2 + c^2 + 2bc$ (–cos A)

$\Rightarrow$ $a^2 = b^2 + c^2 - 2bc \cos A$

6. ABC is a triangle.

The coordinates of A, B and C are (4, 0, 8), (2, -3, 5) and (3, 0, 7) respectively.

$$\overrightarrow{AB} = \text{P.V. of B} - \text{P.V. of A}$$

$$= \left(3\hat{i} + 0\hat{j} + 7\hat{k}\right) - \left(2\hat{i} - 3\hat{j} + 5\hat{k}\right)$$

$$= \hat{i} + 3\hat{j} + 2\hat{k}.$$

$\overrightarrow{AC}$ = P.V. of C – P.V of A

$$= \left(4\hat{i} + 0\hat{j} + 6\hat{k}\right) - \left(2\hat{i} - 3\hat{j} + 5\hat{k}\right)$$

$$= 2\hat{i} + 3\hat{j} + \hat{k}.$$

Vector area of $\Delta ABC = \frac{1}{2}\overrightarrow{AB} \times \overrightarrow{AC}$

$$= \frac{1}{2}\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 3 & 2 \\ 2 & 3 & 1 \end{vmatrix}$$

$$= \frac{1}{2}\left[\hat{i}(3-6) - \hat{j}(1-4) + \hat{k}(3-6)\right]$$

$$= \frac{1}{2}\left(-3\hat{i} + 3\hat{j} - 3\hat{k}\right)$$

$$= \frac{3}{2}\left(-\hat{i} + \hat{j} - \hat{k}\right)$$

Area of $\Delta ABC = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$

$$= \frac{3}{2}\left|-\hat{i} + \hat{j} - \hat{k}\right|$$

$= \frac{3}{2}\sqrt{1+1+1} = \frac{3\sqrt{3}}{2}$ sq. unit.

$$\cos\angle BAC = \frac{\overrightarrow{AB}.\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}$$

$$= \frac{\left(\hat{i} + 3\hat{j} + 2\hat{k}\right).\left(2\hat{i} + 3\hat{j} + \hat{k}\right)}{\sqrt{1+9+4}\sqrt{4+9+1}}$$

$$= \frac{2+9+2}{\sqrt{14}\sqrt{10}} = \frac{13}{\sqrt{140}}$$

$$\Rightarrow \quad m\angle BAC = \cos^{-1}\left(\frac{13}{\sqrt{140}}\right)$$

7. Given that $\vec{a} = 2\hat{i} - 3\hat{j} + 4\hat{k}$

$\vec{b} = \hat{i} - 2\hat{j} - \hat{k}$

$\vec{c} = 3\hat{i} - \hat{j} + 2\hat{k}$

Volume of the parallelopeped = $\left[\vec{a}\vec{b}\vec{c}\right]$

$$= \begin{vmatrix} 2 & -3 & 4 \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix}$$

$$= 2(4-1) - (-3)(2+3) + 4(-1-6)$$

= 6 + 15 – 28

= 21 – 28 = –7 = 7

cubic unit in magnitude

$$\vec{a}\times\vec{b}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -3 & 4\\ 1 & 2 & -1\end{vmatrix}$$

$= \hat{i}(3-8)-\hat{j}(-2-4)+\hat{k}(4+3)$

$= 5\hat{i}+6\hat{j}+7\hat{k}$

$$(\vec{a}\times\vec{b})\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ -5 & 6 & 7\\ 3 & -1 & 2\end{vmatrix}$$

$= \hat{i}(12+7)-\hat{j}(-10-21)+\hat{k}(5-18)$

$= 19\hat{i}+31\hat{j}-13\hat{k}$

8. Given that $\vec{\alpha}=4\hat{i}+5\hat{j}-\hat{k}$

$\vec{\beta}=\hat{i}-\hat{j}+5\hat{k}$

$\vec{q}=3\hat{i}+\hat{j}-\hat{k}$.

Given that $\vec{p}$ is perpendicular to $\vec{a}$ and $\vec{\beta}$.

$\vec{p}$ is parallel to $\vec{a}\times\vec{\beta}$.

$\therefore\quad \vec{p}=\lambda(\vec{\alpha}\times\vec{\beta})$ ......(1)

$$\vec{a}\times\vec{\beta}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 4 & 5 & -1\\ 1 & -4 & 5\end{vmatrix}$$

$= 21\hat{i}-21\hat{j}-21\hat{k}$

From (1), we have

$\vec{p}=\lambda(21\hat{i}-21\hat{j}-21\hat{k})$

$= 21\lambda\hat{i}-21\lambda\hat{j}-21\lambda\hat{k}$ ....(2)

Again given that $\vec{p}.\vec{q}=21$

$\Rightarrow (21\lambda\hat{i}-21\lambda\hat{j}-21\lambda\hat{k}).(3\hat{i}+\hat{j}-\hat{k})=21$

$\Rightarrow 63\lambda-21\lambda+21\lambda=21$

$\Rightarrow 63\lambda=21$

$\Rightarrow \lambda=\frac{21}{63}=\frac{1}{3}$

From (2), we have

$\vec{p}=21.\frac{1}{3}\hat{i}-21.\frac{1}{3}\hat{j}-21.\frac{1}{3}\hat{k}$

$= 7\hat{i}-7\hat{j}-7\hat{k}$.

9. The given lines are

$\frac{x+3}{2}=\frac{y+5}{3}=\frac{2-7}{-3}=r_1$ (say) ......(1)

$\frac{x+1}{4}=\frac{y+1}{5}=\frac{2+1}{-1}=r_2$ (say) ......(2)

Any point on the line (1) is

$(2r_1-3,\ 3r_1-1,\ -r_2-1)$

It two lines are coplanar, they must intersect.

At the point of intersection

$2r_1-3=4r_2-1$

$3r_1-5=5r_2-1$

$-3r_1+7=-r_2-1$

$\Rightarrow 2r_1-4r_2-2=0$ .....(3)

$3r_1-5r_2-4=0$ ....(4)

$3r_1-r_2-8=0$ ......(5)

Solving (3) and (4), we get

$r_1=3$ and $r_2=1$

These values of r_1 and r_2 satisfy the equation(5). So the lines are coplanar.

Euation of the plane containing the lines (1) & (2) is

$$\begin{vmatrix}x+3 & y+5 & z-7\\ 2 & 3 & -3\\ 4 & 5 & -1\end{vmatrix}\begin{vmatrix}x+3 & y+5 & z-7\\ 2 & 3 & -3\\ 4 & 5 & -1\end{vmatrix}=0$$

$\Rightarrow (x+3)(-3+15)-(y+5)(-2+12)$

$+ (2-7)(10-12)=0$

$\Rightarrow 6x-5y-3=0.$

10. Four given points Rae (0, 4, 3), (–1, –5, –3), (–2, –2, 1) and (1,1, –1).

$$\begin{vmatrix} x_4 - x_1 & y_4 - y_1 & z_4 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{vmatrix}$$

$$= \begin{vmatrix} 1-0 & 1-4 & -1-3 \\ -1-0 & -5-4 & -3-3 \\ -2-0 & -2-4 & 1-3 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -3 & -4 \\ -1 & -9 & -6 \\ -2 & -6 & -2 \end{vmatrix}$$

$$= 18 - 36 - (-3)(2-12) + (-4)(6-18)$$

$$= -18 + 3(-10) - 4(-12)$$

$$= -18 - 30 + 48 = 0$$

So four points are coplanar.

Equation of the plane containing the first three points is

$$\begin{vmatrix} x-0 & y-4 & 2-3 \\ -1-0 & -5-4 & -3-3 \\ -2-0 & -2-4 & 1-3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & y-4 & z-3 \\ -1 & -9 & -6 \\ -2 & -6 & -2 \end{vmatrix} = 0$$

$$\Rightarrow x(18-36) - (y-4)(2-12) + (2-3)(6-18) = 0$$

$\Rightarrow$ 9x – 10y + 12z 4 = 0.

11. Let the equation of the plane

$$\ell x + my + nz = 3r \quad ....(1)$$

Where $\ell^2 + m^2 + n^2 = 1$ ....(2)

Let the plane intersects x-axis at A, y-axis at B and z-axis at C.

At A, y = 0, z = 0

From (1), we have $\ell x + mo + no = 3r$

$$\Rightarrow x = \frac{3r}{e}$$

$$\Rightarrow \text{A is } \left(\frac{3r}{e}, 0, 0\right).$$

Similarly B is $\left(0, \frac{3r}{m}, 0\right)$ and C is $\left(0, 0 \frac{3C}{n}\right)$

Let (x, y, z) be the centroid of the ΔABC.

$$\therefore \quad x = \frac{\frac{3r}{e} + 0 + 0}{3} = \frac{r}{e}$$

$$y = \frac{r}{m}, \qquad z = \frac{r}{n}$$

$$\therefore \quad e = \frac{r}{x}, m = \frac{r}{y}, n = \frac{r}{z}$$

From (2), we get

$$\frac{r^2}{x^2} + \frac{r^2}{y^2} + \frac{r^2}{z^2} = 1$$

$$\Rightarrow r^2\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 1$$

$$\Rightarrow \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{r^2}$$

$$\Rightarrow x^{-2} + y^{-2} + z^{-2} = r^{-2}.$$

12. Let ABCDEFG be a rectangular parallelopiped.

Here OA=a, OB=b, OE=c.

Let us take O is origin, OA along x-axis, OC along y-axis and OE along z-axis.

The coordinates of the corner points are O(o,o,o), A(a,o,o), B(a, b, o), C(o, b, o), D(o,b,c), E(o,o,c), F(a,o,c) & G(a,b,c).

The d.rs of OG are $\langle a-o, b-o, c-o\rangle = \langle a, b, c\rangle$.

The d.rs of EB are $\langle a-o, b-o, o-c\rangle = \langle a, b, -c\rangle$.

Let θ be the angle between OG and EB.

$$\cos\theta = \frac{a.a + b.b + c(-c)}{\sqrt{a^2+b^2+c^2}\sqrt{a^2+b^2+(-c)^2}}$$

$$= \frac{a^2+b^2-c^2}{a^2+b^2+c^2}$$

$$\Rightarrow \theta = \cos^{-1}\left(\frac{a^2+b^2-c^2}{a^2+b^2+c^2}\right)$$

Similarly we can find the angle between the other diagonals.

So the angle between the diagonals are

$$\cos^{-1}\left(\frac{\pm a^2 \pm b^2 \pm c^2}{a^2+b^2+c^2}\right)$$

13. Let OA and OB be two mutually perpendicular lines, whose d-cs are $\langle \ell_1, m_1, n_1\rangle$ and $\langle \ell_2, m_2, n_2\rangle$ respectively.

Let OC be a line which is perpendicular to both OA and OB.

Let the d.cs of OC be $\langle \ell, m, n\rangle$

Since OC is perpendicular to OA and OB,

we have $\ell\ell_1 + mm_1 + nn_1 = 0$

$$\ell\ell_2 + mm_2 + nn_2 = 0$$

By cross multiplication, we get

$$\frac{\ell}{m_1n_1 = m_2n_1} = \frac{m}{n_1\ell_2 - n_2\ell_1} = \frac{n}{\ell_1m_2 - \ell_2m_1}$$

$$= \frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{(m_1n_2-m_2n_1)^2+(n_1\ell_2-n_2\ell_1)^2+(\ell_1m_2-\ell_2m_1)^2}}$$

$$= \frac{1}{\sin 90} = 1$$

$$\Rightarrow \ell = m_1n_2 - m_2n_1$$

$$m = n_1\ell_2 - n_2\ell_1$$

$$n = \ell_1n_2 - \ell_2m_1$$

The d.cs of OC are

$$\langle m_1n_2 - m_2n_1,\ n_1\ell_2 - n_2\ell_1, \ell_1m_2 - \ell_2m_1\rangle$$

14. Given plane is

$$2x + 6y + 6z - 1 = 0 \qquad ....(1)$$

Let A and B be two given points whose coordination are (2, 2, 1) and (9, 3, 6).

Equation of the plane passing through (2,2,1) is

$$a(x-2) + b(y-2) + c(z-1) = 0 \qquad ....(1)$$

Since it is passing through (9, 3, 6), we have

$$a(9-2) + b(3-2) + c(6-1) = 0$$

$$\Rightarrow 7a + b + 5c = 0 \qquad .....(2)$$

Since the plane is perpendicular to the plane (1), we have $2a + 6b + 6c = 0$ ......(3)

From (2) & (3) by cross multiplication, we get

$$\frac{a}{6-30} = \frac{b}{10-42} = \frac{c}{42-2}$$

$$\Rightarrow \frac{a}{-24} = \frac{b}{-32} = \frac{c}{40}$$

$$\Rightarrow \frac{a}{3} = \frac{b}{4} = \frac{c}{-5} = k \text{ (say)}$$

$\therefore$ a = 3k, b = 4k, c = –5k

From (2), the required plane is

$$3k(x-2) + 4k(y-2) - 5k(z-1) = 0$$

$$\Rightarrow 3(x-2) + 4(y-2) - 5(z-1) = 0$$

15. Equation of the line passing through (1,2,3) and (2,1,-1) is

$$\frac{x-1}{2-1}=\frac{y-2}{1-2}=\frac{z-3}{-1-3}$$

$$\Rightarrow \quad \frac{x-1}{1}=\frac{b-2}{-1}=\frac{z-3}{-4} \qquad ....(1)$$

Equation of the line passing through (–1,3,1) and (3,1,5) is

$$\frac{x+1}{3+1}=\frac{y-3}{1-3}=\frac{z-1}{5-1}$$

$$\Rightarrow \quad \frac{x+1}{4}=\frac{y-3}{-2}=\frac{z-1}{4} \qquad ......(2)$$

We shall test the coplanerity of the lines (1) & (2).

$$\begin{vmatrix} -1-1 & 3-2 & 1-3 \\ 1 & -1 & -4 \\ 4 & -2 & 4 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 & -2 \\ 1 & -1 & -4 \\ 4 & -2 & 4 \end{vmatrix}$$

$$= -2(-4-8)-1(4+16)+(-2)(-2+4)$$

$$= -2\times(-12)-20-2.2$$

$$= 24-20-4=0$$

So the lines (1) & (2) are coplanar.

The d.rs of the lines are $\langle 1,-1,-4\rangle$ and $\langle 4,-2,4\rangle$.

$\Rightarrow$ The lines are not parallel.

So two lines intersect.

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