Contributions to the Theory of Partitions and their Applications

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CONTRIBUTIONS TO THE THEORY OF PARTITIONS AND THEIR APPLICATIONS

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By

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under the supervision of

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Dedicated To

Beloved my parents and family members

DECLARATION

I hereby declare that the thesis entitled **"Contributions to the theory of partitions and their applications,"** being submitted to the Bangalore University for the degree of **"Doctor of Philosophy"** is the result of the research work carried out by me in the Department of Mathematics, Central College Campus, Bangalore University, Bengaluru under the guidance of Dr M. S. Mahadeva Naika, Professor, Department of Mathematics, Central College Campus, Bangalore University, Bengaluru.

I further declare that the results of this work have not been submitted previously to this or any other University or Institution for any degree.

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CERTIFICATE

This is to certify that the thesis entitled "**Contributions to the theory of partitions and their applications**," being submitted to Bangalore University, for the award of the degree of Doctor of Philosophy in Mathematics is a record of research work carried out by **Mr Shivaprasada Nayaka S.** under my supervision and guidance during the period 2015-2018.

I further certify that no part of this thesis have been submitted elsewhere for award of any other degree.

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List of publications

Papers published

1. M. S. Mahadeva Naika and **S. Shivaprasada Nayaka**, Congruences for $(2, 3)$- regular partition with designated summands, Note Mate., 36(2): 99–123, 2016.

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Papers communicated

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2. M. S. Mahadeva Naika and **S. Shivaprasada Nayaka**, Congruences for partition quadruples with t-cores.

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Chapter 1

INTRODUCTION

1.1 Definitions and Study of Literature

1.1.1 Partitions

A partition of a positive integers n is a finite non-increasing sequence of positive integers $\nu_1 \geq \nu_2 \cdots \geq \nu_m > 0$ such that

$$n = \sum_{i=1}^{m} \nu_i,$$

where ν_i's are called parts. The number of partitions of n is denoted by $p(n)$ and by convention $p(0) = 1$. For example, partitions of 5 are

$$5, \quad 4+1, \quad 3+2, \quad 3+1+1, \quad 2+2+1, \quad 2+1+1+1, \quad 1+1+1+1+1.$$

Thus $p(5) = 7$.

Ferrers Diagram

A Ferrers diagram is a way to represent partitions geometrically. The diagram consists rows of dots. Each row represents a different addend in the partition. The rows are ordered in non-increasing order so that the row with the most dots is on the top and the row with the least dots is on the bottom.

For example: 13 can be partitioned into $5+3+2+2+1$ which would be represented by the following Ferrers diagram:

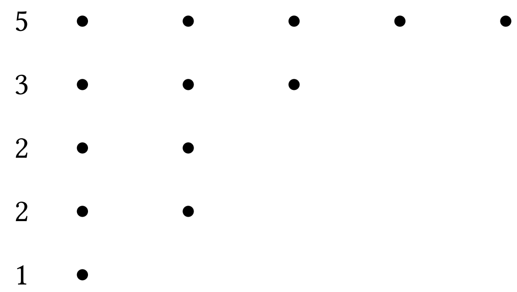

Figure 1.1: Ferrers diagram of the partition $5 + 3 + 2 + 2 + 1$.

The **conjugate** of a Ferrers diagram is formed by reflecting the diagram across its diagonal (the one starting in the top left of the diagram). This can also be interpreted as exchanging the rows for the columns. For example, consider our example from before but this time let's count the number of dots in each column:

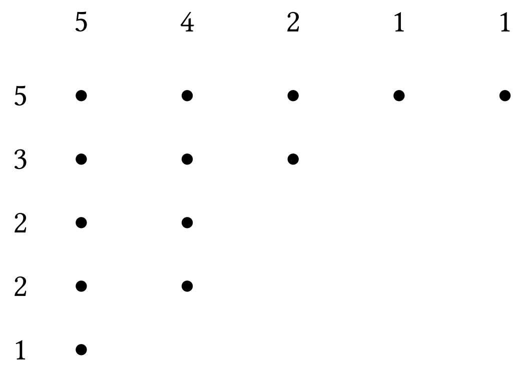

Figure 1.2: Ferrers diagram of the conjugate partition $5 + 4 + 2 + 1 + 1$.

A French mathematician, Philip Naude (1684–1747), raised a number of questions in his letter to Leonhard Euler (1707–1783) in 1740. One of his questions was as follows: in how many ways can an integer n be represented as a sum of integers? In response to this question, Euler discovered many ideas, results and methods of partitions of numbers. These elementary, but remarkable, results were presented in his fundamental treatise on analysis, *Introductio in Analysin Infinitorum* [24]. His fundamental works on the theory of partitions of number based on the use of generating functions and formal power series firmly established the additive number theory.

1.1.2 Generating functions and Notation

Euler gave a generating function for $p(n)$ using the q- series

$$\sum_{n=0}^{\infty} p(n)q^n = \frac{1}{(q;q)_\infty}. \tag{1.1}$$

Now

$$\begin{aligned}
\frac{1}{(q;q)_\infty} &= \frac{1}{1-q} \cdot \frac{1}{1-q^2} \cdot \frac{1}{1-q^3} \cdots \\
&= (1 + q^1 + q^{1+1} + q^{1+1+1} + \cdots)(1 + q^2 + q^{2+2} + \cdots)(1 + q^3 + q^{3+3} + \cdots)\cdots \\
&= 1 + q^1 + q^{1+1} + q^2 + q^{1+1+1} + q^{1+2} + q^3 + \cdots \\
&= 1 + q^1 + (q^{1+1} + q^2) + (q^{1+1+1} + q^{1+2} + q^3) + \cdots \\
&= 1 + q + 2q^2 + 3q^3 + \cdots . \qquad (1.2)
\end{aligned}$$

For any complex number a and q with $|q| < 1$, we have

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1 - aq^k)$$

and for any positive integer k,

$$f_k := (q^k;q^k)_\infty.$$

Euler noted that the series representation of infinite product $(q;q)_\infty$ is given by

$$\begin{aligned}
(q;q)_\infty &= \sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2+k)/2} \\
&= 1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + + - - \cdots . \qquad (1.3)
\end{aligned}$$

The above identity is known as Euler's *pentagonal number* theorem. From (1.1) and (1.3), we have

$$\left(\sum_{n=0}^{\infty} p(n)q^n\right)(1 - q - q^2 + q^5 + q^7 - q^{12} - q^{15} + + - - \cdots) = 1. \tag{1.4}$$

Which implies to get the following recurrence relation:

$$p(n) = p(n-1) + p(n-2) - p(n-5) - p(n-7) + p(n-12) + p(n-15) - \cdots. \quad (1.5)$$

A British mathematician, Percy Alexander MacMahon in 1916, was the first person who computed $p(n)$ for n up to 200, using the Euler's recurrence relation and he made a table with columns of five numbers such as

1	7	42	176	627 ...
1	11	56	231	792 ...
2	15	77	297	1002 ...
3	22	101	385	1255 ...
5	30	135	490	1575 ...

Ramanujan noticed the table and found three simple congruences satisfied by $p(n)$ are as follows: For every nonnegative integer n,

$$p(5n+4) \equiv 0 \pmod 5, \quad (1.6)$$

$$p(7n+5) \equiv 0 \pmod 7, \quad (1.7)$$

$$p(11n+6) \equiv 0 \pmod{11}. \quad (1.8)$$

In [64, 65], Ramanujan gave a proof of above three congruences. He made a remarks in [64] that "It appears that there are no equally simple properties for any moduli involving primes other than 5, 7 and 11". In a posthumously published papers [66] and [67], Hardy has collected different proof of (1.6)-(1.8) from an unpublished manuscript of Ramanujan on $p(n)$ and $\tau(n)$ ([68]).

Ramanujan [65] has noticed a more general conjecture. Let $\zeta = 5^a 7^b 11^c$ and let κ be an integer such that $24\kappa \equiv 1 \pmod{\zeta}$. Then

$$p(\zeta n + \kappa) \equiv 0 \pmod{\zeta}. \quad (1.9)$$

In [68], Ramanujan gave a proof of (1.9) for arbitrary a and $b = c = 0$. He also sketch a proof of his conjecture for arbitrary b and $a = c = 0$, but he did not complete it. After Ramanujan died, H. Gupta extended MacMahon's table up to $n = 300$. Chowla [19] after observing the Gupta's table, found that $p(243)$ is not divisible by 7^3, despite the fact that $24 \cdot 243 \equiv 1 \pmod{7^3}$. To correct Ramanujan's conjecture, define $\zeta' =$

$5^a 7^{b'} 11^c$, where $b' = b$, if $b = 0, 1, 2$, and $b' = [(b + 2)/2]$, if $b > 2$. Then

$$p(\zeta' n + \kappa) \equiv 0 \pmod{\zeta'}. \tag{1.10}$$

Watson [73] published a proof of (1.10) for $a = c = 0$ and noticed a more detailed version of Ramanujan's proof of (1.10) in case $b = c = 0$. Finally, Atkin [6] proved (1.10) for arbitrary c and $a = b = 0$.

We study several congruence properties of restricted partition functions such as: k-color overpartition functions, Andrews' singular overpartitions, Designated summands, ℓ-regular cubic partition pairs, (ℓ, m)-regular bipartition triples and Partition quadruple with t-cores.

A bipartition of a positive integer n is a pair of partitions (ν_1, ν_2) such that the sum of all the parts is equal to n, where ν_1 and ν_2 are allowed to be empty partition. Let $p_{-2}(n)$ denote the number of bipartitions of n. The generating function for $p_{-2}(n)$ is given by

$$\sum_{n=0}^{\infty} p_{-2}(n) q^n = \frac{1}{(q;q)_\infty^2} = \frac{1}{f_1^2}. \tag{1.11}$$

Atkin [7] has proved the Ramanujan type congruences for $p_{-2}(n)$ modulo 5. Ramanathan [63] has established congruences modulo 5 for $p_{-2}(n)$ which are analogues to the classical congruences of Ramanujan.

Let $p_k(n)$ be two color partition function with one of the color is multiple of k, the generating function is given by

$$\sum_{n=0}^{\infty} p_k(n) q^n = \frac{1}{(q;q)_\infty (q^k;q^k)_\infty} = \frac{1}{f_1 f_k}. \tag{1.12}$$

Ahmed, Baruah and Dastidar [2] have found some interesting congruences modulo 5 for $p_k(n)$ for $k \in \{2, 3, 4\}$. Chern [18] has established some congruences modulo 7 for $p_4(n)$. Tang [70] has proved some infinite families of Ramanujan-type congruences modulo powers of 5 for $p_k(n)$ with $k = 2, 6, 7$.

Corteel and Lovejoy [20] have introduced the combinatorial object known as overpartition of a nonnegative integer n, which is a non-increasing sequence of a natural number, whose sum is n and the first (equivalently, the final) occurrence of parts of each size may be over lined. We denote the number of overpartitions of n by $\overline{p}(n)$ and

$\overline{p}(0) = 1$. As noted in [20], the generating function for $\overline{p}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{p}(n)q^n = \frac{(q^2;q^2)_\infty}{(q;q)_\infty^2} = \frac{f_2}{f_1^2}. \tag{1.13}$$

For example: The eight overpartitions of 3 are

$$3, \quad \overline{3}, \quad 2+1, \quad \overline{2}+1, \quad 2+\overline{1}, \quad \overline{2}+\overline{1}, \quad 1+1+1, \quad \overline{1}+1+1.$$

Mahlburg [47] has proved that $\overline{p}(n)$ is divisible by 64 for almost all positive integers n. He also conjectured that for a particular positive integer k, $\overline{p}(n)$ is divisible by 2^k for almost all positive integers n. Kim [38] has proved the $k = 7$ case of the conjecture by using the techniques of Mahlburg. Hirschhorn and Sellers [34] have established some congruence modulo small powers of 2 for $\overline{p}(n)$ and also proved 2-, 3- and 4-dissections of the generating function for overpartition function.

In chapter (2), we establish several infinite families of congruences modulo powers of 2 and 3 for $\overline{p}_3(n)$, where $\overline{p}_3(n)$ denote the number of overpartitions of n with 2-color in which one of the colors appears only in parts that are multiples of 3.

For any positive integer $\ell > 1$, a partition is said to be ℓ-regular if none of its parts is divisible by ℓ. Let $d_\ell(n)$ denote the number of such partitions of n, with $d_\ell(0) = 1$. The generating function for $d_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} d_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty}{(q;q)_\infty} = \frac{f_\ell}{f_1}. \tag{1.14}$$

Many authors have obtained several infinite families of congruences satisfied by $d_\ell(n)$. See [4, 13, 21, 25, 37, 50, 55, 58, 74, 79].

Let $B_\ell(n)$ denote the number of ℓ-regular bipartitions of n. The generating function for $B_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} B_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty^2}{(q;q)_\infty^2} = \frac{f_\ell^2}{f_1^2}. \tag{1.15}$$

Lin [43, 44] has proved infinite family of congruences modulo 3 for $B_7(n)$ using Ramanujan's two modular equations of degree 7. Mahadeva Naika and Hemanthkumar [50] have established several infinite families of congruences modulo powers of 2 and 5 for $B_5(n)$.

Andrews, Lewis and Lovejoy [5] have investigated a new class of partition with designated summands, are constructed by taking ordinary partitions and tagging exactly one of each part size. The total number of partitions of n with designated summands is denoted by $PD(n)$. The authors [5] have derived the following generating function of $PD(n)$:

$$\sum_{n=0}^{\infty} PD(n)q^n = \frac{(q^6;q^6)_\infty}{(q;q)_\infty(q^2;q^2)_\infty(q^3;q^3)_\infty} = \frac{f_6}{f_1 f_2 f_3}. \tag{1.16}$$

For example: $PD(4) = 10$, namely

$$4', \quad 3'+1', \quad 2'+2, \quad 2+2', \quad 2'+1'+1, \quad 2'+1+1', \quad 1'+1+1+1, \quad 1+1'+1+1,$$
$$1+1+1'+1, \quad 1+1+1+1'$$

Andrews et al. [5] and Baruah and Ojah [10] have also studied $PDO(n)$, the number of partitions of n with designated summands in which all parts are odd and the generating function is given by

$$\sum_{n=0}^{\infty} PDO(n)q^n = \frac{f_4 f_6^2}{f_1 f_3 f_{12}}. \tag{1.17}$$

In chapter (3), we obtain several infinite families of congruences modulo 3, 4, 8, 16 and 32 for $PD_{2,3}(n)$, where $PD_{2,3}(n)$ denote the number of partitions of n with designated summands in which parts are not multiples of 2 or 3. Also establish several congruences modulo 3 and 4 for $PBD_3(n)$, where $PBD_3(n)$ denote the number of 3-regular bipartitions of n with designated summands.

Andrews [3] introduced singular overpartitions. To introduce singular overpartitions, first he defined some properties of the entries in a Frobenius symbol for n, which is of the form

$$\begin{pmatrix} a_1 & a_2 & ..., & a_r \\ b_1 & b_2 & ..., & b_r \end{pmatrix},$$

where $\sum(a_i + b_i + 1) = n$ and $a_1 > a_2 > ... > a_r \geq 0, b_1 > b_2 > ... > b_r \geq 0$. There is a natural mapping that reveals a one-to-one correspondence between the Frobenius symbols for n and the ordinary partitions of n. "Singular overpartitions" are Frobenius symbols for n with at most one overlined entry in each row. More precisely, for two positive integers k and i, a column $\begin{pmatrix} a_j \\ b_j \end{pmatrix}$ in a Frobenius symbol is (k,i)–positive if $a_j - b_j \geq k - i - 1$ and (k,i)–negative if $a_j - b_j \leq -i + 1$. If $-i + 1 < a_j - b_j < k - i + 1$, then we say the column is (k,i)–neutral. Two columns have the same parity if they are both (k,i)–positive or (k,i)–negative. We can divide the Frobenius symbol into (k,i)–blocks such that all the entries in each block have either the same (k,i)–parity or (k,i)–neutral. The first non-neutral column in each parity block is called the anchor of the block. A (k,i)–parity block is neutral if all columns in it are neutral and a (k,i)–parity block is positive (resp. negative) if it contains no (k,i)–negative (resp. positive) columns.

A Frobenius symbol is (k,i)–singular if

(1) there are no overlined entries, or

(2) the one overlined entry on the top row occurs in the anchor of a (k,i)–positive block, or

(3) the one overlined entry on the bottom row occurs in an anchor of a (k,i)–negative block, and

(4) if there is one overlined entry in each row, then they occur in adjacent (k,i)–parity blocks.

Let $\overline{Q}_{k,i}(n)$ denote the number of such singular overpartitions for $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$. Andrews proved that $\overline{Q}_{k,i}(n) = \overline{C}_{k,i}(n)$, where $\overline{C}_{k,i}(n)$ counts the number of overpartitions of n in which no part is divisible by k and only parts congruent to $\pm i$ modulo k may be overlined. Therefore for $k \geq 3$ and $1 \leq i \leq \lfloor \frac{k}{2} \rfloor$, the generating function for $\overline{C}_{k,i}(n)$ is given by

$$\sum_{n=0}^{\infty} \overline{Q}_{k,i}(n) q^n = \sum_{n=0}^{\infty} \overline{C}_{k,i}(n) q^n$$
$$= \frac{(q^k;q^k)_\infty(-q^i;q^k)_\infty(-q^{k-i};q^k)_\infty}{(q;q)_\infty}. \tag{1.18}$$

For example: Ten singular overpartitions counted by $\overline{C}_{3,1}(4)$ are

$4,\ \overline{4},\ 2+2,\ \overline{2}+2,\ 2+1+1,\ \overline{2}+1+1,\ 2+\overline{1}+1,\ \overline{2}+\overline{1}+1,\ 1+1+1+1,\ \overline{1}+1+1+1.$

Andrews [3] proves that, for all $n \geq 0$, $\overline{C}_{3,1}(n) = \overline{A}_3(n)$, where $\overline{A}_3(n)$ is the number of overpartitions of n into parts not divisible by 3. The function $\overline{A}_\ell(n)$, which counts the number of overpartitions of n into parts not divisible by ℓ, plays a key role in the work of Lovejoy [45].

In [3], Andrews found the following congruences:

$$\overline{C}_{3,1}(9n+3) \equiv \overline{C}_{3,1}(9n+6) \equiv 0 \pmod{3}. \tag{1.19}$$

In chapter (4), we establish several infinite families of congruences for $\overline{CO}_{3,1}(n)$ modulo 6, 8 and 16, where $\overline{CO}_{\delta,i}(n)$ denote the number of singular overpartitions of

n into odd parts. We obtain congruence and infinite families of congruences modulo 4 for $\overline{A}^{3}_{4,1}(n)$ and modulo 8 for $\overline{A}^{5}_{4,1}(n)$, where $\overline{A}^{k}_{\delta,i}(n)$ denote the number of singular overpartitions of n without multiples of k. Also we deduce some new infinite families of congruences for $\overline{C}^{6}_{1,2}(n)$ modulo 27 and congruences modulo 4 for $\overline{C}^{12}_{1,5}(n)$, $\overline{C}^{9}_{3,3}(n)$ and $\overline{C}^{15}_{5,5}(n)$, where $\overline{C}^{\delta}_{i,j}(n)$ denote the number of Andrews' singular overpatition pairs of n.

Kim [39] studied overcubic partition function $\overline{a}(n)$, which is analogous to overpartition function

$$\sum_{n=0}^{\infty} \overline{a}(n)q^n = \frac{(-q;q)_\infty(-q^2;q^2)_\infty}{(q;q)_\infty(q^2;q^2)_\infty} = \frac{f_4}{f_1^2 f_2}. \tag{1.20}$$

Hirschhorn [28] has obtained the results of Kim [39] using Jacobi's triple product identity. Sellers [69] has proved a number of arithmetic properties of $\overline{a}(n)$. Zhao and Zhong [82] have studied cubic partition pairs denoted by $b(n)$ and the generating function is

$$\sum_{n=0}^{\infty} b(n)q^n = \frac{1}{(q;q)_\infty^2(q^2;q^2)_\infty^2} = \frac{1}{f_1^2 f_2^2}. \tag{1.21}$$

In chapter (5), we establish some infinite families of congruences modulo 4, 8, 27 and 81 for $b_\ell(n)$, where $b_\ell(n)$ denote the number of ℓ- regular cubic partition pairs of a positive integer n and the values of $\ell \in \{2, 3, 5, 9\}$.

A partition k-tuple of n is a k-tuple of partitions $(\nu_1, \nu_2, \ldots, \nu_k)$ such that $n = |\nu_1| + \ldots + |\nu_k|$. We will call a partition 2-tuple a bipartition and a partition 3-tuple a partition triple. A partition triple (ν_1, ν_2, ν_3) of a positive integer n is called ℓ-regular partition triple if none of ν_i, $i = 1, 2$ and 3, is divisible by ℓ. The number of ℓ-regular partition triple of positive integer n is denoted by $T_\ell(n)$. The generating function for $T_\ell(n)$ is given by

$$\sum_{n=0}^{\infty} T_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty^3}{(q;q)_\infty^3} = \frac{f_\ell^3}{f_1^3}. \tag{1.22}$$

Wang [71,72] has established infinite families of arithmetic properties and congruences for overpartition triples and partition triples with 3-cores.

A (ℓ, m)-regular bipartition of n is a bipartition (ν_1, ν_2) of n such that ν_1 is ℓ-regular partition and ν_2 is a m-regular partition. Let $B_{\ell,m}(n)$ denote the number of (ℓ, m)-regular bipartitions of n. The generating function for $B_{\ell,m}(n)$ is

$$\sum_{n=0}^{\infty} B_{\ell,m}(n)q^n = \frac{(q^\ell;q^\ell)_\infty(q^m;q^m)_\infty}{(q;q)_\infty^2} = \frac{f_\ell f_m}{f_1^2}. \tag{1.23}$$

Lin [42–44] has proved several infinite families of congruences modulo 3 for $B_{4,4}(n)$, $B_{7,7}(n)$ and $B_{13,13}(n)$ and gave characterizations of $B_{4,4}(n)$ modulo 2 and 4. Dai [22] examined the behavior of $B_{4,4}(n)$ modulo 8 and found several infinite families of congruences modulo 8 for $B_{4,4}(n)$.

In chapter (6), we obtain some arithmetic identities and congruences modulo 3, 9 and 27 for $BT_{\ell,m}(n)$, where $BT_{\ell,m}(n)$ denote the number of (ℓ,m)-regular bipartition triples of a positive integer n, here $(\ell,m)\in\{(2,9),(3,3),(3,5),(3,7),(3,9),(9,9)\}$.

The Ferrers-Young diagram of the partition ν of n is obtained by arranging n nodes in k left aligned rows so that the i^{th} row has ν_i nodes. The nodes are labeled by row and column coordinates as one would label the entries of a matrix. Let ν_j' denote the number of nodes in column j. The hook number $H(i,j)$ of the (i,j) node is defined as the number of nodes directly below and to the right of the node including the node itself. i. e. $H(i,j)=\nu_i+\nu_j'-j-i+1$. A t-core is a partition with no hook number that are divisible by t.

For example: In Figure (1.1) represents the Ferrers-Young diagram of the partition $\nu=(5,3,2,2,1)$ of 13. The nodes (1, 1), (1, 2), (1, 3), (1, 4), (1,5), (2, 1), (2, 2), (2, 3), (3, 1), (3, 2), (4, 1), (4, 2) and (5, 1) have hook numbers 9, 7, 4, 2, 1, 6, 4, 1, 4, 2, 3, 1 and 1, respectively. Therefore ν is a t-core partition for $t=5$ and for all $t\geq 10$.

Let $a_t(n)$ be the number of partitions of n that are t-cores, then its generating function is given by [[29], Eq. (2.1)]

$$\sum_{n=0}^{\infty} a_t(n)q^n=\frac{(q^t;q^t)_\infty^t}{(q;q)_\infty}=\frac{f_t^t}{f_1}. \tag{1.24}$$

Garvan, Kim and Stanton [26] have proved that if α is a positive integer and $\ell=5,7,11$, then

$$a_\ell(\ell^\alpha n-\delta_\ell)\equiv 0 \pmod{\ell^\alpha}$$

for all nonnegative integer n, where $\delta_\ell=\frac{\ell^2-1}{24}$. Kolitsch and Sellers [41], Hischhorn and Sellers [33] have established parity results for $a_8(n)$ and $a_{16}(n)$. Granville and Ono [27] have obtained Ramanujan type congruences for $a_t(n)$, when t is power of 5, 7 or 11.

A partition k-tuple of n with t-cores is a partition k-tuple $(\nu_1,\nu_2,...,\nu_k)$ of n where each ν_i is t-core for $i=1,2,3,...k$. If $C_t(n)$ denotes the number of partition quadruple of n with t-cores, then the generating function for $C_t(n)$ is given by

$$\sum_{n=0}^{\infty} C_t(n)q^n = \frac{(q^t;q^t)_\infty^{4t}}{(q;q)_\infty^4} = \frac{f_t^{4t}}{f_1^4}. \tag{1.25}$$

In chapter (7), we establish several infinite families of congruences modulo 8 for $C_3(n)$, congruences modulo 5 and 7 for $C_5(n)$, $C_7(n)$ and $C_{25}(n)$.

1.1.3 Ramanujan's theta functions

Ramanujan's general theta function $f(x,y)$ is defined as

$$f(x,y) := \sum_{n=-\infty}^{\infty} x^{n(n+1)/2} y^{n(n-1)/2}, \quad |xy| < 1. \tag{1.26}$$

The product representation of $f(x,y)$ arises from Jacobi's triple product identity [11, p. 35, Entry 19] as

$$f(x,y) = (-x;xy)_\infty(-y;xy)_\infty(xy;xy)_\infty. \tag{1.27}$$

Special cases of $f(x,y)$ are

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2(q^2;q^2)_\infty = \frac{f_2^5}{f_1^2 f_4^2}, \tag{1.28}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{f_2^2}{f_1} \tag{1.29}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty = f_1. \tag{1.30}$$

1.2 Preliminaries

In this section, we collect few results which are useful in proving our main results.

Lemma 1.2.1. *For each prime p and $n \geq 1$,*

$$f_1^{p^n} \equiv f_p^{p^{n-1}} \pmod{p^n}. \tag{1.31}$$

It easily follows from the binomial theorem.

Lemma 1.2.2. *[11, pp. 40-49] We have*

$$\varphi(q) = \varphi(q^9) + 2qf(q^3, q^{15}), \tag{1.32}$$

$$\psi(q) = f(q^3, q^6) + q\psi(q^9). \tag{1.33}$$

Lemma 1.2.3. *[67, p. 212] We have the following* 5*-dissection*

$$f_1 = f_{25}\left(a - q - q^2/a\right), \tag{1.34}$$

where

$$a := a(q) := \frac{(q^{10}, q^{15}; q^{25})_\infty}{(q^5, q^{20}; q^{25})_\infty}.$$

Lemma 1.2.4. *[11, p. 303, Entry 17(v)] We have*

$$f_1 = f_{49}\left(\frac{B(q^7)}{C(q^7)} - q\frac{A(q^7)}{B(q^7)} - q^2 + q^5\frac{C(q^7)}{A(q^7)}\right), \tag{1.35}$$

where

$$A(q) := \frac{f(-q^3, -q^4)}{f(-q^2)}, B(q) := \frac{f(-q^2, -q^5)}{f(-q^2)} \text{and} C(q) := \frac{f(-q, -q^6)}{f(-q^2)}.$$

Lemma 1.2.5. *[21, Theorem 2.2] For any prime $p \geq 5$,*

$$f_1 = \sum_{\substack{k=\frac{1-p}{2} \\ k \neq \frac{\pm p-1}{6}}}^{\frac{p-1}{2}} (-1)^k q^{\frac{3k^2+k}{2}} f\left(-q^{\frac{3p^2+(6k+1)p}{2}}, -q^{\frac{3p^2-(6k+1)p}{2}}\right) + (-1)^{\frac{\pm p-1}{6}} q^{\frac{p^2-1}{24}} f_{p^2}, \tag{1.36}$$

where

$$\frac{\pm p - 1}{6} := \begin{cases} \frac{p-1}{6}, & \text{if } p \equiv 1 \pmod 6, \\ \frac{-p-1}{6}, & \text{if } p \equiv -1 \pmod 6. \end{cases}$$

Furthermore, for $\frac{-(p-1)}{2} \leq k \leq \frac{p-1}{2}$ and $k \neq \frac{(\pm p-1)}{6}$,

$$\frac{3k^2 + k}{2} \not\equiv \frac{p^2 - 1}{24} \pmod p.$$

Lemma 1.2.6. *[21, Theorem 2.1] For any odd prime p,*

$$\psi(q) = \sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2}). \tag{1.37}$$

Furthermore, $\frac{m^2+m}{2} \not\equiv \frac{p^2-1}{8} \pmod{p}$ *for* $0 \le m \le \frac{p-3}{2}$.

Lemma 1.2.7. *The following 2-dissections hold:*

$$f_1^2 = \frac{f_2 f_8^5}{f_4^2 f_{16}^2} - 2q\frac{f_2 f_{16}^2}{f_8}, \tag{1.38}$$

$$\frac{1}{f_1^2} = \frac{f_8^5}{f_2^5 f_{16}^2} + 2q\frac{f_4^2 f_{16}^2}{f_2^5 f_8}, \tag{1.39}$$

$$f_1^4 = \frac{f_4^{10}}{f_2^2 f_8^4} - 4q\frac{f_2^2 f_8^4}{f_4^2}, \tag{1.40}$$

$$\frac{1}{f_1^4} = \frac{f_4^{14}}{f_2^{14} f_8^4} + 4q\frac{f_4^2 f_8^4}{f_2^{10}}. \tag{1.41}$$

Lemma (1.2.7) is a consequence of dissection formulas of Ramanujan, collected in Berndt's book [11, p. 40, Entry 25].

Lemma 1.2.8. *The following 2-dissections hold:*

$$\frac{f_3^3}{f_1} = \frac{f_4^3 f_6^2}{f_2^2 f_{12}} + q\frac{f_{12}^3}{f_4}, \tag{1.42}$$

$$\frac{f_3}{f_1^3} = \frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q\frac{f_4^2 f_6 f_{12}^2}{f_2^7}, \tag{1.43}$$

$$\frac{f_1}{f_3^3} = \frac{f_2 f_4^2 f_{12}^2}{f_6^7} - q\frac{f_2^3 f_{12}^6}{f_4^2 f_6^9}, \tag{1.44}$$

$$\frac{f_1^3}{f_3} = \frac{f_4^3}{f_{12}} - 3q\frac{f_2^2 f_{12}^3}{f_4 f_6^2}. \tag{1.45}$$

Hirschhorn, Garvan and Borwein [29] have proved the equation (1.42). For proof of (1.43), see [9]. Proofs of equations (1.44) and (1.45) follow by changing q to $-q$ in equations (1.42) and (1.43) respectively with $(-q;-q)_\infty = \dfrac{f_2^3}{f_1 f_4}$.

Lemma 1.2.9. *The following* 2*-dissection holds:*

$$\frac{f_3}{f_1} = \frac{f_4 f_6 f_{16} f_{24}^2}{f_2^2 f_8 f_{12} f_{48}} + q\frac{f_6 f_8^2 f_{48}}{f_2^2 f_{16} f_{24}}. \tag{1.46}$$

Xia and Yao [81] gave a proof of Lemma (1.2.9).

Lemma 1.2.10. *The following* 2*-dissections hold:*

$$\frac{f_3^2}{f_1^2} = \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}}, \tag{1.47}$$

$$\frac{f_1^2}{f_3^2} = \frac{f_2 f_4^2 f_{12}^4}{f_6^5 f_8 f_{24}} - 2q\frac{f_2^2 f_8 f_{12} f_{24}}{f_4 f_6^4}. \tag{1.48}$$

Xia and Yao [78] proved (1.47) and (1.48) follows from (1.47).

Lemma 1.2.11. *The following* 2*-dissection formulas hold:*

$$f_1 f_3 = \frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_6 f_{24}^2} - q\frac{f_4^4 f_6 f_{24}^2}{f_2 f_8^2 f_{12}^2}, \tag{1.49}$$

$$\frac{1}{f_1 f_3} = \frac{f_8^2 f_{12}^5}{f_2^2 f_4 f_6^4 f_{24}^2} + q\frac{f_4^5 f_{24}^2}{f_2^4 f_6^2 f_8^2 f_{12}}, \tag{1.50}$$

$$\frac{1}{f_1^2 f_3^2} = \frac{f_8^5 f_{24}^5}{f_2^5 f_6^5 f_{16}^2 f_{48}^2} + 2q\frac{f_4^4 f_{12}^4}{f_2^6 f_6^6} + 4q^4\frac{f_4^2 f_{12}^2 f_{16}^4 f_{48}^2}{f_2^5 f_6^5 f_8 f_{24}}. \tag{1.51}$$

Baruah and Ojah [10] have proved last Lemma.

Lemma 1.2.12. *The following* 2*-dissections hold:*

$$\frac{f_5}{f_1} = \frac{f_8 f_{20}^2}{f_2^2 f_{40}} + q\frac{f_4^3 f_{10} f_{40}}{f_2^3 f_8 f_{20}}, \tag{1.52}$$

$$\frac{f_1}{f_5} = \frac{f_2 f_8 f_{20}^3}{f_4 f_{10}^3 f_{40}} - q\frac{f_4^2 f_{40}}{f_8 f_{10}^2}. \tag{1.53}$$

Equation (1.52) was proved by Hirschhorn and Sellers [30]. Replacing q by $-q$ in (1.52), we obtain (1.53).

Lemma 1.2.13. *The following 2-dissections hold:*

$$\frac{1}{f_1 f_7} = \frac{f_{16}^2 f_{56}^5}{f_2^2 f_8 f_{14}^2 f_{28}^2 f_{112}^2} + q\frac{f_4^2 f_{28}^2}{f_2^3 f_{14}^3} + q^6 \frac{f_8^5 f_{112}^2}{f_2^2 f_4^2 f_{14}^2 f_{16}^2 f_{56}}, \tag{1.54}$$

$$f_1 f_7 = \frac{f_2 f_{14} f_{16}^2 f_{56}^5}{f_4 f_8 f_{28}^3 f_{112}^2} - q f_4 f_{28} + q^6 \frac{f_2 f_8^5 f_{14} f_{112}^2}{f_4^3 f_{16}^2 f_{28} f_{56}}. \tag{1.55}$$

Equation (1.54) and (1.55) was proved by Xia and Yao [76].

Lemma 1.2.14. *The following 2-dissections hold:*

$$\frac{f_9}{f_1} = \frac{f_{12}^3 f_{18}}{f_2^2 f_6 f_{36}} + q\frac{f_4^2 f_6 f_{36}}{f_2^3 f_{12}}, \tag{1.56}$$

$$\frac{f_1}{f_9} = \frac{f_2 f_{12}^3}{f_4 f_6 f_{18}^2} - q\frac{f_4 f_6 f_{36}^2}{f_{12} f_{18}^3}. \tag{1.57}$$

Lemma 1.2.14 was proved by Xia and Yao [77].

Lemma 1.2.15. *[50, Lemma 2.3] The following 2-dissection formulas hold:*

$$f_1 f_5^3 = 2q^2 f_4 f_{20}^3 + f_2^3 f_{10} - 2q^3 \frac{f_4^4 f_{40}^2 f_{10}}{f_2 f_8^2} - q\frac{f_2^2 f_{10}^2 f_{20}}{f_4}, \tag{1.58}$$

$$f_1^3 f_5 = 2q^2 \frac{f_4^6 f_{40}^2 f_{10}}{f_2 f_8^2 f_{20}^2} + \frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_4^3 f_{20} - 5q f_2 f_{10}^3. \tag{1.59}$$

Ramanujan's cubic continued fraction ω is given by

$$\omega := \frac{q^{1/3}}{1} \; \underset{+}{} \; \frac{q+q^2}{1} \; \underset{+}{} \; \frac{q^2+q^4}{1} \; \underset{+}{} \; \frac{q^3+q^6}{1} \; \underset{+}{} \; \cdots, \tag{1.60}$$

We define the function $x(q)$, $a(q)$, $b(q)$ and $c(q)$ as follows:

$$\begin{aligned} x(q) &= q^{-1/3}\omega, \\ a(q) &= \frac{1}{x(q)^2} - 2qx(q), \\ b(q) &= \frac{1}{x(q)} + 4qx(q)^2, \\ c(q) &= \frac{1}{x(q)^3} - 8q^2 x(q)^3. \end{aligned}$$

From the definition of $a(q)$, $b(q)$ $c(q)$, we get the following results

$$a(q)b(q) = c(q) + 2q, \tag{1.61}$$
$$a(q)^3 + qb(q)^3 = c(q)^2 - 5qc(q) + 40q^2. \tag{1.62}$$

Lemma 1.2.16. *[14] We have*

$$f_1 f_2 = f_9 f_{18}\left(\frac{1}{x(q^3)} - q - 2q^2 x(q^3)\right), \tag{1.63}$$
$$\frac{1}{f_1 f_2} = \frac{f_9^3 f_{18}^3}{f_3^4 f_6^4}\left(a(q^3) + qb(q^3) + 3q^2\right), \tag{1.64}$$
$$c(q) = \frac{f_1^4 f_2^4}{f_3^4 f_6^4} + 7q. \tag{1.65}$$

Lemma 1.2.17. *Let*

$$\sum_{n=0}^{\infty} h(n)q^n = \frac{1}{f_1^4 f_2^4}. \tag{1.66}$$

Then

$$\sum_{n=0}^{\infty} h(3n+2)q^n \equiv 18\frac{f_3^4 f_6^4}{f_1^8 f_2^8} + 81q\frac{f_3^8 f_6^8}{f_1^{12} f_2^{12}} \pmod{243}. \tag{1.67}$$

Proof. Consider

$$\sum_{n=0}^{\infty} h(n)q^n = \frac{1}{f_1^4 f_2^4}. \tag{1.68}$$

Employing (1.64) into (1.68), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} h(n)q^n =& \frac{f_9^{12} f_{18}^{12}}{f_3^{16} f_6^{16}}(a(q^3)^4 + 4qa(q^3)^3 b(q^3) + 6q^2 a(q^3)^2 b(q^3)^2 \\ &+ 12q^2 a(q^3)^3 + 36q^3 a(q^3)^2 b(q^3) + 4q^3 a(q^3) b(q^3)^3 \\ &+ q^4 b(q^3)^4 + 36q^4 a(q^3) b(q^3)^2 + 54q^4 a(q^3)^2 \\ &+ 108q^5 a(q^3) b(q^3) + 12q^5 b(q^3)^3 + 54q^6 b(q^3)^2 \\ &+ 108q^6 a(q^3) + 108q^7 b(q^3) + 81q^8) \pmod{243}.\end{aligned} \tag{1.69}$$

Extracting the terms involving q^{3n+2} from (1.69), dividing q^2 and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} h(3n+2)q^n = \frac{f_3^{12} f_6^{12}}{f_1^{16} f_2^{16}} (6a(q)^2 b(q)^2 + 12(a(q)^3 + qb(q)^3) + 108qa(q)b(q) + 81q^2) \pmod{243}. \tag{1.70}$$

In view of (1.61), (1.62) and (1.70), it follows that

$$\sum_{n=0}^{\infty} h(3n+2)q^n \equiv \frac{f_3^{12} f_6^{12}}{f_1^{16} f_2^{16}} (18c(q)^2 + 72qc(q) + 72q^2) \pmod{243}. \tag{1.71}$$

Substituting (1.65) into (1.71), we arrive at (1.67). □

Lemma 1.2.18. *The following 3-dissection holds:*

$$\frac{f_2}{f_1^2} = \frac{f_6^4 f_9^6}{f_3^8 f_{18}^3} + 2q \frac{f_6^3 f_9^3}{f_3^7} + 4q^2 \frac{f_6^2 f_{18}^3}{f_3^6}. \tag{1.72}$$

Equation (1.72) was proved by Hirschhorn and Sellers [35].

Lemma 1.2.19. *[10, Lemma 2.6] The following 3-dissection formula holds:*

$$\frac{f_4}{f_1} = \frac{f_{12} f_{18}^4}{f_3^3 f_{36}^2} + q \frac{f_6^2 f_9^3 f_{36}}{f_3^4 f_{18}^2} + 2q^2 \frac{f_6 f_{18} f_{36}}{f_3^3}. \tag{1.73}$$

Lemma 1.2.20. *[36] The following 3-dissection formula holds:*

$$f_1 f_2 = \frac{f_6 f_9^4}{f_3 f_{18}^2} - q f_9 f_{18} - 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2}. \tag{1.74}$$

Lemma 1.2.21. *[11, p. 345, Entry 1 (iv)] We have the following 3-dissection*

$$f_1^3 = f_9^3 (\zeta^{-1} - 3q + 4q^3 \zeta^2), \tag{1.75}$$

where

$$\zeta = \frac{f_3 f_{18}^3}{f_6 f_9^3}.$$

Lemma 1.2.22. *[11, p. 345, Entry 1] We have*

$$\frac{f_1^{12}}{f_3^{12}} + 27q = (\eta^{-1} + 4q\eta^2)^3, \tag{1.76}$$

where

$$\eta := \frac{f_1 f_6^3}{f_2 f_3^3}.$$

Lemma 1.2.23. *[12] The following* 3*-dissection holds:*

$$\frac{f_1}{f_4} = \frac{f_6 f_9 f_{18}}{f_{12}^3} - q\frac{f_3 f_{18}^4}{f_{12}^3 f_9^2} - q^2\frac{f_6^2 f_9 f_{36}^3}{f_{12}^4 f_{18}^2}. \tag{1.77}$$

Let $p \geq 3$ be a prime and a be an integer. The Legendre symbol $\left(\frac{a}{p}\right)$ is defined by

$$\left(\frac{a}{p}\right) := \begin{cases} 1, & \text{if } a \text{ is a quadratic residue modulo } p \text{ and } p \nmid a, \\ 0, & \text{if } p \mid a, \\ -1, & \text{if } a \text{ is a quadratic nonresidue modulo } p. \end{cases}$$

Chapter 2

2-COLOR OVERPARTITION FUNCTION

2.1 Introduction

In the introductory chapter, we have defined the k-color partition function $p_k(n)$. Lovejoy and Mallet [46] have defined the basic notions associated with n-color overpartitions and also determined some basic generating functions. Motivated by above works, we define,

$$\sum_{n=0}^{\infty} \overline{p}_3(n)q^n = \frac{(-q;q)_\infty(-q^3;q^3)_\infty}{(q;q)_\infty(q^3;q^3)_\infty}. \tag{2.1.1}$$

Let $\overline{p}_3(n)$ denote the number of overpartitions of n with 2-color in which one of the colors appears only in parts that are multiples of 3. For example, there are ten partitions of 2-color overpartitions of 3:

$$3_a,\ \overline{3}_a,\ 3_b,\ \overline{3}_b,\ 2_a+1_a,\ \overline{2}_a+1_a,\ 2_a+\overline{1}_a,\ \overline{2}_a+\overline{1}_a,\ 1_a+1_a+1_a,\ \overline{1}_a+1_a+1_a.$$

2.2 Infinite families of congruences for 2-color overpartitions

In this section, we establish several infinite families of congruences modulo powers of 2 and 3 for $\overline{p}_3(n)$.

References [59] is belongs to this chapter

2.2.1 Congruences modulo 9 and 18

Theorem 2.2.1. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(12 \cdot 2^{\alpha} n) \equiv \overline{p}_3(6n) \pmod{9}, \tag{2.2.1}$$

$$\overline{p}_3(4 \cdot 3^{\alpha+3} n + 2 \cdot 3^{\alpha+3}) \equiv \overline{p}_3(36n+18) \pmod{9}, \tag{2.2.2}$$

$$\overline{p}_3(12n+6) \equiv 6 \cdot \overline{p}_3(6n+3) \pmod{9}, \tag{2.2.3}$$

$$\overline{p}_3(108n+18) \equiv \overline{p}_3(36n+6) \pmod{9}, \tag{2.2.4}$$

$$\overline{p}_3(36n+30) \equiv 0 \pmod{9}, \tag{2.2.5}$$

$$\overline{p}_3(6n+4) \equiv 0 \pmod{18}. \tag{2.2.6}$$

Proof. We have

$$\sum_{n=0}^{\infty} \overline{p}_3(n) q^n = \frac{f_2 f_6}{f_1^2 f_3^2}. \tag{2.2.7}$$

Substituting (1.72) in (2.2.7), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(n) q^n = \frac{f_6^5 f_9^6}{f_3^{10} f_{18}^3} + 2q \frac{f_6^4 f_9^3}{f_3^9} + 4q^2 \frac{f_6^3 f_{18}^3}{f_3^8}. \tag{2.2.8}$$

Extracting the terms involving q^{3n+1}, dividing by q and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+1) q^n = 2 \frac{f_2^4 f_3^3}{f_1^9}. \tag{2.2.9}$$

Invoking (1.31) into (2.2.9), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+1) q^n \equiv 2 f_2^4 \pmod{18}. \tag{2.2.10}$$

Extracting the terms involving q^{2n+1} from (2.2.10), we obtain (2.2.6).

From (2.2.8), we have

$$\sum_{n=0}^{\infty} \overline{p}_3(3n) q^n = \frac{f_2^5 f_3^6}{f_1^{10} f_6^3}. \tag{2.2.11}$$

Invoking (1.31) into (2.2.11), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_3^3}{f_1 f_2^4} \pmod{9}. \tag{2.2.12}$$

Employing (1.42) into (2.2.12), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_4^3 f_6^2}{f_2^6 f_{12}} + q\frac{f_{12}^3}{f_2^4 f_4} \pmod{9}. \tag{2.2.13}$$

Extracting the terms involving q^{2n} from (2.2.13) and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(6n)q^n \equiv \frac{f_2^3 f_3^2}{f_1^6 f_6} \pmod{9}. \tag{2.2.14}$$

Invoking (1.31) into (2.2.14), we find that

$$\sum_{n=0}^{\infty} \overline{p}_3(6n)q^n \equiv \frac{f_1^3 f_2^3}{f_3 f_6} \pmod{9}. \tag{2.2.15}$$

Employing (1.45) into (2.2.15), we have

$$\sum_{n=0}^{\infty} \overline{p}_3(6n)q^n \equiv \frac{f_2^3 f_4^3}{f_6 f_{12}} + 6q\frac{f_2^5 f_{12}^3}{f_4 f_6^3} \pmod{9}. \tag{2.2.16}$$

Extracting the terms involving q^{2n+1} from (2.2.16), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+6)q^n \equiv 6\frac{f_1^5 f_6^3}{f_2 f_3^3} \pmod{9}. \tag{2.2.17}$$

Invoking (1.31) into (2.2.17), we get

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+6)q^n \equiv 6\frac{f_1^2 f_6^3}{f_2 f_3^2} \pmod{9}. \tag{2.2.18}$$

Replacing q by $-q$ in (1.32) and using the fact that

$$\phi(-q) = \frac{f_1^2}{f_2}, \tag{2.2.19}$$

we find that

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}. \tag{2.2.20}$$

Again employing (2.2.20) into (2.2.18), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+6)q^n \equiv 6\frac{f_6^3 f_9^2}{f_3^2 f_{18}} - 12q\frac{f_6^2 f_{18}^2}{f_3 f_9} \pmod 9. \tag{2.2.21}$$

Congruence (2.2.5) follows by extracting the terms involving q^{3n+2} on both sides of (2.2.21).

Extracting the terms involving q^{3n+1} from (2.2.21) and dividing by q, then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(36n+18)q^n \equiv 6\frac{f_2^2 f_6^2}{f_1 f_3} \pmod 9. \tag{2.2.22}$$

It follows from (1.33) that

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{2.2.23}$$

Employing (2.2.23) into (2.2.22), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(36n+18)q^n \equiv 6\frac{f_6^3 f_9^2}{f_3^2 f_{18}} + 6q\frac{f_6^2 f_{18}^2}{f_3 f_9} \pmod 9. \tag{2.2.24}$$

Extracting the terms involving q^{3n+1} from (2.2.24) and dividing by q, then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(108n+54)q^n \equiv 6\frac{f_2^2 f_6^2}{f_1 f_3} \pmod 9. \tag{2.2.25}$$

In view of congruences (2.2.22) and (2.2.25), we have

$$\overline{p}_3(108n+54) \equiv \overline{p}_3(36n+18) \pmod 9. \tag{2.2.26}$$

Utilizing (2.2.26) and by mathematical induction on α, we get (2.2.2).

Extracting the terms involving q^{3n} from (2.2.21) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(36n+6)q^n \equiv 6\frac{f_2^3 f_3^2}{f_1^2 f_6} \pmod 9. \tag{2.2.27}$$

Extracting the terms involving q^{3n} from (2.2.24) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(108n+18)q^n \equiv 6\frac{f_2^3 f_3^2}{f_1^2 f_6} \pmod{9}. \tag{2.2.28}$$

In view of congruences (2.2.27) and (2.2.28), we arrive at (2.2.4).

Extracting the terms involving q^{2n} from both sides of (2.2.16) and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(12n)q^n \equiv \frac{f_1^3 f_2^3}{f_3 f_6} \pmod{9}. \tag{2.2.29}$$

In view of congruences (2.2.15) and (2.2.29), we have

$$\overline{p}_3(12n) \equiv \overline{p}_3(6n) \pmod{9}. \tag{2.2.30}$$

Utilizing (2.2.30) and by mathematical induction on α, we arrive at (2.2.1).

Extracting the terms involving q^{2n+1} from (2.2.13) and dividing by q, then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+3)q^n \equiv \frac{f_6^3}{f_1^4 f_2} \pmod{9}. \tag{2.2.31}$$

Extracting the terms involving q^{2n+1} from (2.2.16) and dividing by q, then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+6)q^n \equiv 6\frac{f_1^5 f_6^3}{f_2 f_3^3} \pmod{9}. \tag{2.2.32}$$

Invoking (1.31) into (2.2.32), we have

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+6)q^n \equiv 6\frac{f_6^3}{f_1^4 f_2} \pmod{9}. \tag{2.2.33}$$

In view of congruences (2.2.31) and (2.2.33), we arrive at (2.2.3). □

2.2.2 Infinite family of congruence modulo 18

Theorem 2.2.2. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(6\cdot 5^{2\alpha+4}n + (30i+25)5^{2\alpha+2}) \equiv 0 \pmod{18}, \tag{2.2.34}$$

where $i = 1, 2, 3, 4$.

Proof. Extracting the terms involving q^{2n} from (2.2.10) and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+1)q^n \equiv 2f_1^4 \pmod{18}. \tag{2.2.35}$$

Employing (1.34) into (2.2.35) and extracting the terms involving q^{5n+4}, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(30n+25)q^n \equiv 8f_5^4 \pmod{18}, \tag{2.2.36}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{p}_3(150n+25)q^n \equiv 8f_1^4 \pmod{18}. \tag{2.2.37}$$

From (2.2.35) and (2.2.37), we find that

$$\overline{p}_3(150n+25) \equiv 4\overline{p}_3(6n+1) \pmod{18}. \tag{2.2.38}$$

Utilizing (2.2.38) and by mathematical induction on α, we obtain

$$\overline{p}_3(6 \cdot 5^{2\alpha+2}n + 5^{2\alpha+2}) \equiv 4^{\alpha+1}\overline{p}_3(6n+1) \pmod{18}. \tag{2.2.39}$$

From (2.2.36), we get

$$\overline{p}_3(150n+30i+25) \equiv 0 \pmod{18}, \quad i = 1, 2, 3, 4. \tag{2.2.40}$$

Using (2.2.39) and (2.2.40), we obtain (2.2.34). □

2.2.3 Congruences modulo 27

Theorem 2.2.3. *For* $\alpha \geq 0$ *and* $n \geq 0$,

$$\overline{p}_3(12n+10) \equiv 0 \pmod{27}, \tag{2.2.41}$$

$$\overline{p}_3(3 \cdot 4^{\alpha+2}n + 10 \cdot 4^{\alpha+1}) \equiv 0 \pmod{27}. \tag{2.2.42}$$

Proof. From (2.2.9), we have

$$\sum_{n=0}^{\infty}\overline{p}_3(3n+1)q^n = 2f_2^4\left(\frac{f_3}{f_1^3}\right)^3. \tag{2.2.43}$$

Employing (1.43) into (2.2.43) and invoking (1.31), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(3n+1)q^n \equiv 2\frac{f_4^{18}f_6^9}{f_2^{23}f_{12}^6} + 18q\frac{f_4^{14}f_6^7}{f_2^{21}f_{12}^2} \pmod{27}. \tag{2.2.44}$$

Extracting the terms involving q^{2n+1} from (2.2.44), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(6n+4)q^n \equiv 18\frac{f_2^{14}f_3^7}{f_1^{21}f_6^2} \pmod{27}. \tag{2.2.45}$$

Invoking (1.31) into (2.2.45), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_3(6n+4)q^n \equiv 18f_2^5f_6 \pmod{27}. \tag{2.2.46}$$

Congruence (2.2.41) follows by extracting the terms involving q^{2n+1} from both sides of (2.2.46).

From (2.2.46), we have

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+4)q^n \equiv 18f_1^5f_3 \pmod{27}. \tag{2.2.47}$$

Using (1.31) into (2.2.47), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+4)q^n \equiv 18f_1^8 \pmod{27}. \tag{2.2.48}$$

Invoking (1.40) into (2.2.48), we find that

$$\sum_{n=0}^{\infty}\overline{p}_3(12n+4)q^n \equiv 18\frac{f_4^{20}}{f_2^4f_8^8} + 18q^2\frac{f_2^4f_8^8}{f_4^4} + 18qf_4^8 \pmod{27}. \tag{2.2.49}$$

Extracting the terms involving q^{2n+1} from (2.2.49), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(24n+16)q^n \equiv 18f_2^8 \pmod{27}. \tag{2.2.50}$$

We can rewrite the above equation as

$$\sum_{n=0}^{\infty} \overline{p}_3(24n+16)q^n \equiv 18f_2^5 f_6 \pmod{27}. \tag{2.2.51}$$

In view of congruences (2.2.46) and (2.2.51), we have

$$\overline{p}_3(6n+4) \equiv \overline{p}_3(24n+16) \pmod{27}. \tag{2.2.52}$$

Utilizing (2.2.52) and by mathematical induction on α, we arrive at

$$\overline{p}_3(6n+4) \equiv \overline{p}_3(6\cdot 4^{\alpha+1}n + 4^{\alpha+2}) \pmod{27}. \tag{2.2.53}$$

Using (2.2.53) and (2.2.41), we get (2.2.42). □

2.2.4 Congruences modulo 8, 16 and 32

Theorem 2.2.4. *For each $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(3^\alpha n) \equiv \overline{p}_3(n) \pmod{8}, \tag{2.2.54}$$

$$\overline{p}_3(18n+6) \equiv 2\overline{p}_3(9n+3) \pmod{8}, \tag{2.2.55}$$

$$\overline{p}_3(3\cdot 4^{\alpha+1}n + 10\cdot 4^\alpha) \equiv 0 \pmod{16}, \tag{2.2.56}$$

$$\overline{p}_3(6\cdot 4^{\alpha+1}n + 5\cdot 4^{\alpha+1}) \equiv 0 \pmod{32}, \tag{2.2.57}$$

$$\overline{p}_3(6n+5) \equiv 0 \pmod{32}, \tag{2.2.58}$$

$$\overline{p}_3(18n+15) \equiv 0 \pmod{8}. \tag{2.2.59}$$

Proof. Invoking (1.31) into (2.2.11), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_2 f_6}{f_1^2 f_3^2} \pmod{8}. \tag{2.2.60}$$

In view of congruences (2.2.7) and (2.2.60), we have

$$\overline{p}_3(3n) \equiv \overline{p}_3(n) \pmod{8}. \tag{2.2.61}$$

Utilizing (2.2.61) and by mathematical induction on α, we arrive at (2.2.54).

Invoking (1.31) into (2.2.9), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+1)q^n \equiv 2\frac{f_3^3}{f_1} \pmod{16}. \tag{2.2.62}$$

Using (1.42) in (2.2.62) and extracting the terms involving q^{2n+1}, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+4)q^n \equiv 2\frac{f_6^3}{f_2} \pmod{16}, \tag{2.2.63}$$

which implies that

$$\overline{p}_3(12n+10) \equiv 0 \pmod{16} \tag{2.2.64}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+4)q^n \equiv 2\frac{f_3^3}{f_1} \pmod{16}. \tag{2.2.65}$$

Using (2.2.62) and (2.2.65), we find that

$$\overline{p}_3(12n+4) \equiv \overline{p}_3(3n+1) \pmod{16}. \tag{2.2.66}$$

By mathematical induction on α, we get

$$\overline{p}_3(3\cdot 4^{\alpha+1}n+4^{\alpha+1}) \equiv \overline{p}_3(3n+1) \pmod{16}. \tag{2.2.67}$$

Congruence (2.2.56) follows from (2.2.64) and (2.2.67).

Equating the terms containing q^{3n+2} from both sides of (2.2.8), dividing by q^2 and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+2)q^n = 4\frac{f_2^3 f_6^3}{f_1^8}. \tag{2.2.68}$$

Invoking (1.31) into (2.2.68), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n+2)q^n \equiv 4\frac{f_6^3}{f_2} \pmod{32}, \tag{2.2.69}$$

which implies,

$$\overline{p}_3(6n+5) \equiv 0 \pmod{32} \tag{2.2.70}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+2)q^n \equiv 4\frac{f_3^3}{f_1} \pmod{32}. \tag{2.2.71}$$

Employing (1.42) into (2.2.71), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+2)q^n \equiv 4\frac{f_4^3 f_6^2}{f_2^2 f_{12}} + 4q\frac{f_{12}^3}{f_4} \pmod{32}. \tag{2.2.72}$$

Extracting the terms involving q^{2n+1} from (2.2.72), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(12n+8)q^n \equiv 4\frac{f_6^3}{f_2} \pmod{32}, \tag{2.2.73}$$

which implies that

$$\overline{p}_3(24n+20) \equiv 0 \pmod{32} \tag{2.2.74}$$

and

$$\sum_{n=0}^{\infty} \overline{p}_3(24n+8)q^n \equiv 4\frac{f_3^3}{f_1} \pmod{32}. \tag{2.2.75}$$

In view of congruences (2.2.71) and (2.2.75), and by mathematical induction on α, we find that

$$\overline{p}_3(6\cdot 4^{\alpha+1}n + 2\cdot 4^{\alpha+1}) \equiv \overline{p}_3(6n+2) \pmod{32}. \tag{2.2.76}$$

Congruence (2.2.57) follows from (2.2.74) and (2.2.76).

Invoking (1.31) into (2.2.11), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_2 f_3^6}{f_1^2 f_6^3} \pmod{8}. \tag{2.2.77}$$

Employing (1.72) into (2.2.77), we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(3n)q^n \equiv \frac{f_6 f_9^6}{f_3^2 f_{18}^3} + 2q\frac{f_9^3}{f_3} + 4q^2\frac{f_{18}^3}{f_6} \pmod{8}. \tag{2.2.78}$$

Extracting the terms involving q^{3n+2} from (2.2.78), dividing by q^2 and replacing q^3 by

q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(9n+6)q^n \equiv 4\frac{f_6^3}{f_2} \pmod{8},$$

which implies,

$$\overline{p}_3(18n+15) \equiv 0 \pmod{8} \tag{2.2.79}$$

and

$$\sum_{n=0}^{\infty}\overline{p}_3(18n+6)q^n \equiv 4\frac{f_3^3}{f_1} \pmod{8}. \tag{2.2.80}$$

Again extracting the terms involving q^{3n+1} from (2.2.78), dividing by q and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(9n+3)q^n \equiv 2\frac{f_3^3}{f_1} \pmod{8}. \tag{2.2.81}$$

Congruence (2.2.55) follows from (2.2.80) and (2.2.81). □

2.2.5 Infinite families of congruences modulo 4

Theorem 2.2.5. *For $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{p}_3(4^{\alpha}n) \equiv \overline{p}_3(n) \pmod{4}, \tag{2.2.82}$$

$$\overline{p}_3(6(4n+i)+1) \equiv 0 \pmod{4}, \tag{2.2.83}$$

where $i = 1, 2, 3$.

$$\overline{p}_3(24\cdot 25^{\alpha+2}n+(120j+25)\cdot 25^{\alpha+1}) \equiv 0 \pmod{4}, \tag{2.2.84}$$

where $j = 1, 2, 3, 4$.

$$\overline{p}_3(2^{2\alpha+2}n+2^{2\alpha+1}) \equiv 0 \pmod{4}, \tag{2.2.85}$$

$$\overline{p}_3(2\cdot 3^{\alpha+2}n+5\cdot 3^{\alpha+1}) \equiv 0 \pmod{16}, \tag{2.2.86}$$

$$\overline{p}_3(6n+5) \equiv 0 \pmod{16}. \tag{2.2.87}$$

Proof. Employing (1.51) into (2.2.7), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(n)q^n=\frac{f_8^5f_{24}^5}{f_2^4f_6^4f_{16}^2f_{48}^2}+2q\frac{f_4^4f_{12}^4}{f_2^5f_6^5}+4q^4\frac{f_4^2f_{12}^2f_{16}^4f_{48}^2}{f_2^4f_6^4f_8f_{24}}. \tag{2.2.88}$$

Extracting the terms involving q^{2n+1} from (2.2.88), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty}\overline{p}_3(2n+1)q^n=2\frac{f_2^4f_6^4}{f_1^5f_3^5}. \tag{2.2.89}$$

Invoking (1.31) into (2.2.89), we deduce that

$$\sum_{n=0}^{\infty}\overline{p}_3(2n+1)q^n\equiv 2f_1^3f_3^3 \pmod{16}. \tag{2.2.90}$$

Employing (1.75) into (2.2.90), we obtain

$$\sum_{n=0}^{\infty}\overline{p}_3(2n+1)q^n\equiv 2\frac{f_3^2f_6f_9^6}{f_{18}^3}+8q^3\frac{f_3^5f_{18}^6}{f_6^2f_9^3}-6qf_3^3f_9^3 \pmod{16}. \tag{2.2.91}$$

Congruence (2.2.87) follows by extracting the terms involving q^{3n+2} on both sides of (2.2.91).

Extracting the terms involving q^{3n+1} from (2.2.91), dividing by q and replacing q^3 by q, we get

$$\overline{p}_3(6n+3)q^n\equiv 10f_1^3f_3^3 \pmod{16}. \tag{2.2.92}$$

Using (2.2.90) and (2.2.92), we have

$$\overline{p}_3(6n+3)\equiv 5\overline{p}_3(2n+1) \pmod{16}. \tag{2.2.93}$$

Utilizing (2.2.93) and by mathematical induction on α, we get

$$\overline{p}_3(6\cdot 3^{\alpha}n+3^{\alpha+1})\equiv 5^{\alpha+1}\overline{p}_3(2n+1) \pmod{16}. \tag{2.2.94}$$

Using (2.2.94) and (2.2.87), we get (2.2.86).

Extracting the terms involving q^{2n} from (2.2.88) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(2n)q^n = \frac{f_4^5 f_{12}^5}{f_1^4 f_3^4 f_8^2 f_{24}^2} + 4q^2 \frac{f_2^2 f_6^2 f_8^4 f_{24}^2}{f_1^4 f_3^4 f_4 f_{12}}, \tag{2.2.95}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{p}_3(2n)q^n \equiv \frac{f_4^5 f_{12}^5}{f_1^4 f_3^4 f_8^2 f_{24}^2} \pmod 4. \tag{2.2.96}$$

Invoking (1.31) into (2.2.96), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(2n)q^n \equiv \frac{f_4 f_{12}}{f_2^2 f_6^2} \pmod 4. \tag{2.2.97}$$

Extracting the terms involving q^{2n+1} from (2.2.97), we obtain

$$\overline{p}_3(4n+2) \equiv 0 \pmod 4. \tag{2.2.98}$$

Again extracting the terms involving q^{2n} from (2.2.97) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(4n)q^n \equiv \frac{f_2 f_6}{f_1^2 f_3^2} \pmod 4. \tag{2.2.99}$$

In view of congruences (2.2.7) and (2.2.99), we have

$$\overline{p}_3(4n) \equiv \overline{p}_3(n) \pmod 4. \tag{2.2.100}$$

Utilizing (2.2.100) and by mathematical induction on α, we get (2.2.82). Using (2.2.98) in (2.2.82), we obtain (2.2.85).

Extracting the terms involving q^{3n} from (2.2.91) and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+1)q^n \equiv 2\frac{f_1^2 f_2 f_3^6}{f_6^3} + 8q\frac{f_1^5 f_6^6}{f_2^2 f_3^3} \pmod{16}, \tag{2.2.101}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+1)q^n \equiv 2\frac{f_1^2 f_2 f_3^6}{f_6^3} \pmod 4. \tag{2.2.102}$$

Invoking (1.31) into (2.2.102), we deduce that

$$\sum_{n=0}^{\infty} \overline{p}_3(6n+1)q^n \equiv 2f_4 \pmod{4}. \tag{2.2.103}$$

Congruence (2.2.83) follows by extracting the terms involving q^{4n+i} on both sides of (2.2.103).

Extracting the terms involving q^{4n} from (2.2.103) and replacing q^4 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(24n+1)q^n \equiv 2f_1 \pmod{4}. \tag{2.2.104}$$

Employing (1.34) into (2.2.104) and extracting the term q^{5n+1}, we obtain

$$\sum_{n=0}^{\infty} \overline{p}_3(120n+25)q^n \equiv 2f_5 \pmod{4}. \tag{2.2.105}$$

Extracting the terms involving q^{5n+i} from (2.2.105), we get

$$\overline{p}_3(600n+120i+25) \equiv 0 \pmod{4}, \quad i = 1,2,3,4. \tag{2.2.106}$$

Extracting the terms involving q^{5n} from (2.2.105) and replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} \overline{p}_3(600n+25)q^n \equiv 2f_1 \pmod{4}. \tag{2.2.107}$$

Using (2.2.104) and (2.2.107), we get

$$\overline{p}_3(600n+25) \equiv \overline{p}_3(24n+1) \pmod{4}. \tag{2.2.108}$$

Utilizing (2.2.108) and by mathematical induction on α, we get

$$\overline{p}_3(600 \cdot 25^{\alpha} n + 25^{\alpha+1}) \equiv \overline{p}_3(24n+1) \pmod{4}. \tag{2.2.109}$$

Utilizing (2.2.106) and (2.2.109), we get (2.2.84). □

Chapter 3

DESIGNATED SUMMANDS

3.1 Introduction

In chapter (1), we defined partition with designated summands $PD(n)$. Chen, Ji, Jin and Shen [17] have established Ramanujan type identity for the partition function $PD(3n+2)$, they also gave a combinatorial interpretation of the congruence for $PD(3n+2)$ by introducing a rank for partitions with designated summands. Recently Xia [75] extended the work of deriving congruence properties of $PD(n)$ by employing the generating functions of $PD(3n)$ and $PD(3n+2)$ due to Chen et al. [17]. Naika et al. [48,60] have found generating function identities and congruences modulo 4, 9, 12, 36, 48 and 144 for $PD_3(n)$ and studied various arithmetic properties of $PD_2(n)$ modulo 3 and powers of 2.

3.2 Congruences for (2, 3)-regular partition with designated summands

In this section, we define $PD_{2,3}(n)$, the number of partitions of n with designated summands in which parts are not multiples of 2 or 3. The generating function of $PD_{2,3}(n)$ is given by

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}. \tag{3.2.1}$$

For example: $PD_{2,3}(4) = 4$, namely

$$1' + 1 + 1 + 1, \quad 1 + 1' + 1 + 1, \quad 1 + 1 + 1' + 1, \quad 1 + 1 + 1 + 1'.$$

References [55] and [58] are based on this chapter

3.2.1 Congruences modulo 4

Theorem 3.2.1. *For $n \geq 1$ and $\alpha \geq 0$,*

$$PD_{2,3}(18n) \equiv 0 \pmod{4}, \tag{3.2.2}$$

$$PD_{2,3}(2 \cdot 3^{\alpha+3} n) \equiv 0 \pmod{4}. \tag{3.2.3}$$

Proof. We have

$$\sum_{n=0}^{\infty} PD_{2,3}(n) q^n = \frac{f_4 f_6^2 f_9 f_{36}}{f_1 f_{12}^2 f_{18}^2}. \tag{3.2.4}$$

Substituting (1.56) into (3.2.4), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(n) q^n = \frac{f_4 f_6 f_{12}}{f_2^2 f_{18}} + q \frac{f_4^3 f_6^3 f_{36}^2}{f_2^3 f_{12}^3 f_{18}^2}. \tag{3.2.5}$$

Extracting the even terms in the above equation

$$\sum_{n=0}^{\infty} PD_{2,3}(2n) q^n = \frac{f_2 f_3 f_6}{f_1^2 f_9}. \tag{3.2.6}$$

Substituting (1.72) into (3.2.6), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(2n) q^n = \frac{f_6^5 f_9^5}{f_3^7 f_{18}^3} + 2q \frac{f_6^4 f_9^2}{f_3^6} + 4q^2 \frac{f_6^3 f_{18}^3}{f_3^5 f_9}. \tag{3.2.7}$$

Extracting the terms involving q^{3n} from both sides of (3.2.7) and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n) q^n = \frac{f_2^5 f_3^5}{f_1^7 f_6^3}. \tag{3.2.8}$$

Invoking (1.31) into (3.2.8), we find that that

$$\sum_{n=0}^{\infty} PD_{2,3}(6n) q^n \equiv \frac{f_1 f_2 f_3^5}{f_6^3} \pmod{8}. \tag{3.2.9}$$

Employing (1.74) into (3.2.9), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_3^4 f_9^4}{f_6^2 f_{18}^2} - q\frac{f_3^5 f_9 f_{18}}{f_6^3} - 2q^2\frac{f_3^6 f_{18}^4}{f_6^4 f_9^2} \pmod{8}. \tag{3.2.10}$$

Extracting the terms involving q^{3n} from both sides of (3.2.10) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_1^4 f_3^4}{f_2^2 f_6^2} \pmod{8}. \tag{3.2.11}$$

Congruence (3.2.2) follows from (1.31) and (3.2.11).

Equation (3.2.11) can be rewritten as

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_3^4}{f_6^2}\left(\frac{f_1^2}{f_2}\right)^2 \pmod{8}. \tag{3.2.12}$$

Replacing q by $-q$ in (1.32) and using the fact that

$$\phi(-q) = \frac{f_1^2}{f_2}, \tag{3.2.13}$$

we find that that

$$\frac{f_1^2}{f_2} = \frac{f_9^2}{f_{18}} - 2q\frac{f_3 f_{18}^2}{f_6 f_9}. \tag{3.2.14}$$

Employing (3.2.14) into (3.2.12), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n)q^n \equiv \frac{f_3^4 f_9^4}{f_6^2 f_{18}^2} + 4q^2\frac{f_3^6 f_{18}^4}{f_6^4 f_9^2} - 4q\frac{f_3^5 f_9 f_{18}}{f_6^3} \pmod{8}. \tag{3.2.15}$$

Extracting the terms involving q^{3n} from both sides of (3.2.15) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(54n)q^n \equiv \frac{f_1^4 f_3^4}{f_2^2 f_6^2} \pmod{8}. \tag{3.2.16}$$

In view of the congruences (3.2.11) and (3.2.16), we get

$$PD_{2,3}(54n) \equiv PD_{2,3}(18n) \pmod{8}. \tag{3.2.17}$$

Utilizing (3.2.17) and by mathematical induction on α, we arrive at

$$PD_{2,3}(2\cdot 3^{\alpha+3}n) \equiv PD_{2,3}(18n) \pmod 8. \tag{3.2.18}$$

Using (3.2.2) into (3.2.18), we get (3.2.3). □

3.2.2 Congruences and Internal congruence modulo 4 and 8

Theorem 3.2.2. *For $n \geq 0$ and $\alpha \geq 0$,*

$$PD_{2,3}(72n+42) \equiv 0 \pmod 4, \tag{3.2.19}$$

$$PD_{2,3}(36n+30) \equiv 0 \pmod 4, \tag{3.2.20}$$

$$PD_{2,3}(144n+120) \equiv 0 \pmod 4, \tag{3.2.21}$$

$$PD_{2,3}(9\cdot 4^{\alpha+3}n+30\cdot 4^{\alpha+2}) \equiv 0 \pmod 4, \tag{3.2.22}$$

$$PD_{2,3}(54n+18) \equiv 4\cdot PD_{2,3}(18n+6) \pmod 8, \tag{3.2.23}$$

$$PD_{2,3}(54n+36) \equiv 2\cdot PD_{2,3}(18n+12) \pmod 8, \tag{3.2.24}$$

$$PD_{2,3}(36n+30) \equiv 2\cdot PD_{2,3}(72n+60) \pmod 8. \tag{3.2.25}$$

Proof. Extracting the terms involving q^{3n+1} from (3.2.15), dividing by q and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(54n+18)q^n \equiv -4\frac{f_1^5 f_3 f_6}{f_2^3} \pmod 8. \tag{3.2.26}$$

Extracting the terms involving q^{3n+1} from (3.2.10), dividing by q and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv -\frac{f_1^5 f_3 f_6}{f_2^3} \pmod 8. \tag{3.2.27}$$

From (3.2.26) and (3.2.27), we arrive at (3.2.23).

Extracting the terms involving q^{3n+2} from (3.2.15), dividing by q^2 and then replacing q^3 by q, we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(54n+36)q^n \equiv 4\frac{f_1^6 f_6^4}{f_2^4 f_3^2} \pmod 8. \tag{3.2.28}$$

Extracting the terms involving q^{3n+2} from (3.2.10), dividing by q^2 and then replacing

q^3 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv -2\frac{f_1^6 f_6^4}{f_2^4 f_3^2} \pmod{8}. \tag{3.2.29}$$

In view of the congruences (3.2.28) and (3.2.29), we get (3.2.24).

From (3.2.27), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv 7\frac{f_1^5 f_3 f_6}{f_2^3} \pmod{8}. \tag{3.2.30}$$

Invoking (1.31) into (3.2.30), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv 7\frac{f_2 f_3 f_6}{f_1^3} \pmod{8}. \tag{3.2.31}$$

Employing (1.43) into (3.2.31), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+6)q^n \equiv 7\frac{f_4^6 f_6^4}{f_2^8 f_{12}^2} + 21q\frac{f_4^2 f_6^2 f_{12}^2}{f_2^6} \pmod{8}. \tag{3.2.32}$$

Extracting the terms involving q^{2n} from (3.2.32) and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+6)q^n \equiv 7\frac{f_2^6 f_3^4}{f_1^8 f_6^2} \pmod{8}. \tag{3.2.33}$$

Invoking (1.31) into (3.2.33), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+6)q^n \equiv 3f_2^2 \pmod{4}. \tag{3.2.34}$$

Extracting the terms involving q^{2n+1} from (3.2.34), we get (3.2.19).

Extracting the terms involving q^{2n+1} from (3.2.32), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5\frac{f_2^2 f_3^2 f_6^2}{f_1^6} \pmod{8}. \tag{3.2.35}$$

Invoking (1.31) into (3.2.35), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5\frac{f_6^2}{f_2^2}(f_1f_3)^2 \pmod{8}. \tag{3.2.36}$$

Employing (1.49) into (3.2.36), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+24)q^n \equiv 5\frac{f_8^4f_{12}^8}{f_4^4f_{24}^4} + 5q^2\frac{f_4^8f_6^4f_{24}^4}{f_2^4f_8^4f_{12}^4} - 10q\frac{f_4^2f_6^2f_{12}^2}{f_2^2} \pmod{8}. \tag{3.2.37}$$

Extracting the terms involving q^{2n+1} from (3.2.37), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+60)q^n \equiv 6\frac{f_2^2f_3^2f_6^2}{f_1^2} \pmod{8}. \tag{3.2.38}$$

Invoking (1.31) into equation (3.2.29), we find that that

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6\frac{f_3^8}{f_1^2f_3^2} \pmod{8}. \tag{3.2.39}$$

Invoking (1.31) into (3.2.39), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 2\frac{f_6^3}{f_2} \pmod{4}. \tag{3.2.40}$$

Congruence (3.2.20) follows by extracting the terms involving q^{2n+1} from (3.2.40).

From (3.2.40),

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+12)q^n \equiv 2\frac{f_3^3}{f_1} \pmod{4}. \tag{3.2.41}$$

Substituting (1.42) into (3.2.41), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+12)q^n \equiv 2\frac{f_4^3f_6^2}{f_2^2f_{12}} + 2q\frac{f_{12}^3}{f_4} \pmod{4}, \tag{3.2.42}$$

which implies,

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+48)q^n \equiv 2\frac{f_6^3}{f_2} \pmod{4}. \tag{3.2.43}$$

Congruence (3.2.21) fellows by extracting the terms involving q^{2n+1} from (3.2.43).

From equations (3.2.43) and (3.2.40), we have

$$PD_{2,3}(72n+48) \equiv PD_{2,3}(18n+12) \pmod{4}. \tag{3.2.44}$$

By mathematical induction on α, we arrive at

$$PD_{2,3}(18\cdot 4^{\alpha+1}+3\cdot 4^{\alpha+2}) \equiv PD_{2,3}(18n+12) \pmod{4}. \tag{3.2.45}$$

Using (3.2.21) into (3.2.45), we get (3.2.22).

Equation (3.2.39) can rewritten as

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6\left(\frac{f_3^3}{f_1}\right)^2 \pmod{8}. \tag{3.2.46}$$

Employing (1.42) into (3.2.46), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(18n+12)q^n \equiv 6\frac{f_4^6 f_6^4}{f_2^4 f_{12}^2} + 6q^2\frac{f_{12}^6}{f_4^2} + 12q\frac{f_4^2 f_6^2 f_{12}^2}{f_2^2} \pmod{8}. \tag{3.2.47}$$

Extracting the terms involving q^{2n+1} from (3.2.47), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+30)q^n \equiv 12\frac{f_2^2 f_3^2 f_6^2}{f_1^2} \pmod{8}. \tag{3.2.48}$$

From (3.2.38) and (3.2.48), we get (3.2.25). □

3.2.3 Congruences modulo 4

Theorem 3.2.3. *For each $n \geq 0$ and $\alpha \geq 0$,*

$$PD_{2,3}(72\cdot 25^{\alpha+1}n+6\cdot 25^{\alpha+1}) \equiv PD_{2,3}(72n+6) \pmod{4}, \tag{3.2.49}$$

$$PD_{2,3}(360(5n+i)+150) \equiv 0 \pmod{4}, \tag{3.2.50}$$

where $i = 1, 2, 3, 4$.

Proof. From the equation (3.2.34), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+6)q^n \equiv 3f_1^2 \pmod{4}. \tag{3.2.51}$$

Employing (1.34) in the above equation, and then extracting the terms containing q^{5n+2}, dividing by q^2 and replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(360n+150)q^n \equiv 3f_5^2 \pmod{4}, \tag{3.2.52}$$

which yields

$$\sum_{n=0}^{\infty} PD_{2,3}(1800n+150)q^n \equiv 3f_1^2 \equiv \sum_{n=0}^{\infty} PD_{2,3}(72n+6)q^n \pmod{4}. \tag{3.2.53}$$

By induction on α, we obtain (3.2.49). The congruence (3.2.50) follows by extracting the terms involving q^{5n+i} for $i = 1, 2, 3, 4$ from both sides of (3.2.52). □

3.2.4 Congruences modulo 16

Theorem 3.2.4. *For each $n \geq 0$ and $\alpha \geq 0$,*

$$PD_{2,3}(24n+20) \equiv 0 \pmod{16}, \tag{3.2.54}$$

$$PD_{2,3}(6 \cdot 4^{\alpha+2}n + 5 \cdot 4^{\alpha+2}) \equiv 0 \pmod{16}. \tag{3.2.55}$$

Proof. Extracting the terms involving q^{3n+1} from (3.2.7), dividing by q and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n = 2\frac{f_2^4 f_3^2}{f_1^6}. \tag{3.2.56}$$

Invoking (1.31) into equation (3.2.56), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n = 2(f_1 f_3)^2 \pmod{16}. \tag{3.2.57}$$

Substituting (1.49) into (3.2.57), we arrive at

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+2)q^n \equiv 2\frac{f_2^2 f_8^4 f_{12}^8}{f_4^4 f_6^2 f_{24}^4} + 2q^2\frac{f_4^8 f_6^2 f_{24}^4}{f_2^2 f_8^4 f_{12}^4} - 4q f_4^2 f_{12}^2 \pmod{16}. \tag{3.2.58}$$

Extracting the terms involving q^{2n+1} from (3.2.58), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+8)q^n \equiv 12 f_2^2 f_6^2 \pmod{16}. \tag{3.2.59}$$

Extracting the terms involving q^{2n+1} from (3.2.59), we get (3.2.54).

Extracting the terms involving q^{2n} from (3.2.59) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+8)q^n \equiv 12(f_1 f_3)^2 \pmod{16}. \tag{3.2.60}$$

In view of the congruences (3.2.57) and (3.2.60), we get

$$PD_{2,3}(24n+8) \equiv 6 \cdot PD_{2,3}(6n+2) \pmod{16}. \tag{3.2.61}$$

Utilizing (3.2.61) and by mathematical induction on α, we arrive at

$$PD_{2,3}(6 \cdot 4^{\alpha+1} + 2 \cdot 4^{\alpha+1}) \equiv 6^{\alpha+1} \cdot PD_{2,3}(6n+2) \pmod{16}. \tag{3.2.62}$$

Using (3.2.54) into (3.2.62), we arrive at (3.2.55). □

Theorem 3.2.5. *For each $n \geq 0$ and $\alpha \geq 0$,*

$$PD_{2,3}(6 \cdot 4^{\alpha+1} n + 4^{\alpha+2}) \equiv PD_{2,3}(6n+4) \pmod{32}. \tag{3.2.63}$$

Proof. Extracting the terms involving q^{3n+2} from (3.2.7), dividing by q^2 and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n = 4\frac{f_2^3 f_6^3}{f_1^5 f_3}. \tag{3.2.64}$$

Invoking (1.31) into (3.2.64), we arrive at

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n \equiv 4\frac{f_1^3 f_6^3}{f_2 f_3} \pmod{32}. \tag{3.2.65}$$

Employing (1.45) into (3.2.65), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+4)q^n \equiv 4\frac{f_4^3 f_6^3}{f_2 f_{12}} - 12q\frac{f_2 f_6 f_{12}^3}{f_4} \pmod{32}. \tag{3.2.66}$$

Extracting the terms involving q^{2n} from (3.2.66) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+4)q^n \equiv 4\frac{f_2^3 f_3^3}{f_1 f_6} \pmod{32}. \tag{3.2.67}$$

Employing (1.42) into (3.2.67), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+4)q^n \equiv 4\frac{f_2 f_4^3 f_6}{f_{12}} + 4q\frac{f_2^3 f_{12}^3}{f_4 f_6} \pmod{32}. \tag{3.2.68}$$

Extracting the terms involving q^{2n+1} from (3.2.68), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+16)q^n \equiv 4\frac{f_1^3 f_6^3}{f_2 f_3} \pmod{32}. \tag{3.2.69}$$

In view of the congruences (3.2.65) and (3.2.69), we obtain

$$PD_{2,3}(24n+16) \equiv PD_{2,3}(6n+4) \pmod{32}. \tag{3.2.70}$$

Utilizing (3.2.70) and by mathematical induction on α, we get (3.2.63). □

3.2.5 Congruences and Infinite families of congruences modulo 8

Theorem 3.2.6. *For $n \geq 0$,*

$$PD_{2,3}(48n+34) \equiv 0 \pmod{8}, \tag{3.2.71}$$

$$PD_{2,3}(48n+46) \equiv 0 \pmod{8}, \tag{3.2.72}$$

$$PD_{2,3}(96n+52) \equiv 0 \pmod{8}, \tag{3.2.73}$$

$$PD_{2,3}(96n+76) \equiv 0 \pmod{8}. \tag{3.2.74}$$

Proof. Extracting the terms involving q^{2n+1} from (3.2.66), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+10)q^n \equiv 20\frac{f_1 f_3 f_6^3}{f_2} \pmod{32}. \tag{3.2.75}$$

Substituting (1.49) into (3.2.75), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+10)q^n \equiv 20\frac{f_6^2 f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - 20q\frac{f_4^4 f_6^4 f_{24}^2}{f_2^2 f_8^2 f_{12}^2} \pmod{32}. \tag{3.2.76}$$

Extracting the terms involving q^{2n} from (3.2.76) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 20\frac{f_3^2 f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{32}. \tag{3.2.77}$$

Invoking (1.31) into (3.2.77), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 4f_2^2 f_3^2 \pmod{16}. \tag{3.2.78}$$

Invoking (1.31) into (3.2.78), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+10)q^n \equiv 4f_4 f_6 \pmod{8}. \tag{3.2.79}$$

Congruence (3.2.71) follows by extracting the terms involving q^{2n+1} from (3.2.79).

Extracting the terms involving q^{2n+1} from (3.2.76), dividing by q and then replacing

q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+22)q^n \equiv 12\frac{f_2^4 f_3^4 f_{12}^2}{f_1^2 f_4^2 f_6^2} \pmod{32}. \tag{3.2.80}$$

Invoking (1.31) into (3.2.80), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+22)q^n \equiv 12\frac{f_3^4 f_6^2}{f_1^2} \pmod{16}. \tag{3.2.81}$$

Invoking (1.31) into (3.2.81), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+22)q^n \equiv 4\frac{f_6^2 f_{12}}{f_2} \pmod{8}. \tag{3.2.82}$$

Extracting the terms involving q^{2n+1} from (3.2.82), we get (3.2.72).

Extracting the terms involving q^{2n} from (3.2.68) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+4)q^n \equiv 4\frac{f_1 f_2^3 f_3}{f_6} \pmod{32}. \tag{3.2.83}$$

Substituting (1.49) into (3.2.83), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+4)q^n \equiv 4\frac{f_2^4 f_8^2 f_{12}^4}{f_4^2 f_6^2 f_{24}^2} - 4q\frac{f_2^2 f_4^4 f_{24}^2}{f_8^2 f_{12}^2} \pmod{32}. \tag{3.2.84}$$

Extracting the terms involving q^{2n} from (3.2.84) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+4)q^n \equiv 4\frac{f_1^4 f_4^2 f_6^4}{f_2^2 f_3^2 f_{12}^2} \pmod{32}. \tag{3.2.85}$$

Invoking (1.31) into (3.2.85), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+4)q^n \equiv 4\frac{f_4^2}{f_3^2} \pmod{16}. \tag{3.2.86}$$

Invoking (1.31) into (3.2.86), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+4)q^n \equiv 4\frac{f_8}{f_6} \pmod{8}. \tag{3.2.87}$$

Congruence (3.2.73) obtained by extracting the term involving q^{2n+1} from (3.2.87).

Extracting the terms involving q^{2n+1} from (3.2.84), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+28)q^n \equiv 28\frac{f_1^2 f_2^4 f_{12}^2}{f_4^2 f_6^2} \pmod{32}. \tag{3.2.88}$$

Invoking (1.31) into (3.2.88), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+28)q^n \equiv 12 f_1^2 f_6^2 \pmod{16}. \tag{3.2.89}$$

Invoking (1.31) into (3.2.89), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+28)q^n \equiv 4 f_2 f_{12} \pmod{8}. \tag{3.2.90}$$

Extracting the terms involving q^{2n+1} from (3.2.90), we get (3.2.74). □

Theorem 3.2.7. *For any prime* $p \equiv 5$, $\alpha \geq 1$ *and* $n \geq 0$,

$$\sum_{n=0}^{\infty} PD_{2,3}(48p^{2\alpha}n+10p^{2\alpha})q^n \equiv 4 f_2 f_3 \pmod{8}. \tag{3.2.91}$$

Proof. Extracting the terms involving q^{2n} from (3.2.79) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+10)q^n \equiv 4 f_2 f_3 \pmod{8}. \tag{3.2.92}$$

Define

$$\sum_{n=0}^{\infty} f(n)q^n = f_2 f_3 \pmod{8}. \tag{3.2.93}$$

Combining (3.2.92) and (3.2.93), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(48n+10)q^n \equiv 4\sum_{n=0}^{\infty} f(n)q^n \pmod 8. \tag{3.2.94}$$

Now, we consider the congruence equation

$$2\cdot\frac{3k^2+k}{2}+3\cdot\frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod p, \tag{3.2.95}$$

which is equivalent to

$$(2\cdot(6k+1))^2+6\cdot(6m+1)^2 \equiv 0 \pmod p,$$

where $\frac{-(p-1)}{2} \le k,m \le \frac{p-1}{2}$ and p is a prime such that $(\frac{-6}{p}) = -1$. Since $(\frac{-6}{p}) = -1$ for $p \equiv 5 \pmod 6$, the congruence relation (3.2.95) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute (1.36) into (3.2.93) and then extracting the terms in which the powers of q are congruent to $5\cdot\frac{p^2-1}{24}$ modulo p and then divide by $q^{5\cdot\frac{p^2-1}{24}}$, we find that

$$\sum_{n=0}^{\infty} f\left(pn+5\cdot\frac{p^2-1}{24}\right)q^{pn} = f_{2p}f_{3p},$$

which implies,

$$\sum_{n=0}^{\infty} f\left(p^2n+5\cdot\frac{p^2-1}{24}\right)q^{n} = f_2f_3 \tag{3.2.96}$$

and for $n \ge 0$,

$$f\left(p^2n+pi+5\cdot\frac{p^2-1}{24}\right) = 0, \tag{3.2.97}$$

where i is an integer and $1 \le i \le p-1$. By induction, we see that for $n \ge 0$ and $\alpha \ge 0$,

$$f\left(p^{2\alpha}n+5\cdot\frac{p^{2\alpha}-1}{24}\right) = f(n). \tag{3.2.98}$$

Replacing n by $p^{2\alpha}n+5\cdot\frac{p^{2\alpha}-1}{24}$ in (3.2.94), we arrive at (3.2.91). □

Corollary 3.2.1. *For each $n \geq 0$ and $\alpha \geq 0$,*

$$PD_{2,3}(3\cdot 4^{\alpha+3}n+34\cdot 4^{\alpha+1}) \equiv 0 \pmod 8, \tag{3.2.99}$$

$$PD_{2,3}(3\cdot 4^{\alpha+3}n+46\cdot 4^{\alpha+1}) \equiv 0 \pmod 8, \tag{3.2.100}$$

$$PD_{2,3}(6\cdot 4^{\alpha+3}n+13\cdot 4^{\alpha+2}) \equiv 0 \pmod 8, \tag{3.2.101}$$

$$PD_{2,3}(6\cdot 4^{\alpha+3}n+19\cdot 4^{\alpha+2}) \equiv 0 \pmod 8. \tag{3.2.102}$$

Proof. Corollary (3.2.1) follows from the Theorem (3.2.5) and Theorem (3.2.6). □

3.2.6 Congruences and Internal congruence modulo 4

Theorem 3.2.8. *For $n \geq 0$,*

$$PD_{2,3}(12n+11) \equiv 0 \pmod 4, \tag{3.2.103}$$

$$PD_{2,3}(24n+19) \equiv 0 \pmod 4, \tag{3.2.104}$$

$$PD_{2,3}(24n+17) \equiv 0 \pmod 4, \tag{3.2.105}$$

$$PD_{2,3}(108n+63) \equiv 0 \pmod 4, \tag{3.2.106}$$

$$PD_{2,3}(108n+99) \equiv 0 \pmod 4, \tag{3.2.107}$$

$$\sum_{n=0}^{\infty} PD_{2,3}(216n+27)q^n \equiv 2\psi(q) \pmod 4, \tag{3.2.108}$$

$$PD_{2,3}(72n+6) \equiv PD_{2,3}(36n+3) \pmod 4, \tag{3.2.109}$$

$$PD_{2,3}(96n+28) \equiv 2\cdot PD_{2,3}(24n+7) \pmod 4. \tag{3.2.110}$$

Proof. Extracting the odd terms in (3.2.5), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n = \frac{f_2^3 f_3^3 f_{18}^2}{f_1^3 f_6^3 f_9^2}. \tag{3.2.111}$$

Invoking (1.31) into (3.2.111), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n \equiv \frac{f_1 f_2 f_3^3 f_9^2}{f_6^3} \pmod 4. \tag{3.2.112}$$

Substituting (1.74) into (3.2.112), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n \equiv \frac{f_3^2 f_9^6}{f_6^2 f_{18}^2} - q\frac{f_3^3 f_9^3 f_{18}}{f_6^3} - 2q^2\frac{f_3^4 f_{18}^4}{f_6^4} \pmod 4. \tag{3.2.113}$$

Extracting the terms involving q^{3n} from (3.2.113) and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_1^2 f_3^6}{f_2^2 f_6^2} \pmod 4. \tag{3.2.114}$$

Invoking (1.31) into (3.2.114), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_3^2}{f_1^2} \pmod 4. \tag{3.2.115}$$

Employing (1.47) into (3.2.115), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+1)q^n \equiv \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \pmod 4. \tag{3.2.116}$$

Extracting the terms involving q^{2n+1} from (3.2.116), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+7)q^n \equiv 2\frac{f_2 f_3^2 f_4 f_{12}}{f_1^4 f_6} \pmod 4. \tag{3.2.117}$$

Invoking (1.31) into (3.2.117), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+7)q^n \equiv 2f_2 f_{12} \pmod 4. \tag{3.2.118}$$

Extracting the terms involving q^{2n+1} from (3.2.118), we obtain (3.2.103).

From (3.2.118), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+7)q^n \equiv 2f_1 f_6 \pmod 4. \tag{3.2.119}$$

Extracting the terms involving q^{2n} from (3.2.90) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(96n+28)q^n \equiv 4f_1f_6 \pmod{8}. \tag{3.2.120}$$

In view of congruences (3.2.120) and (3.2.119), we obtain (3.2.110).

Extracting the terms involving q^{3n+1} from (3.2.113), dividing by q and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+3)q^n \equiv 3\frac{f_1^3f_3^3f_6}{f_2^3} \pmod{4}. \tag{3.2.121}$$

Invoking (1.31) into (3.2.121), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+3)q^n \equiv 3\frac{f_3^3f_6}{f_1f_2} \pmod{4}. \tag{3.2.122}$$

Employing (1.42) into (3.2.122), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+3)q^n \equiv 3\frac{f_4^3f_6^3}{f_2^3f_{12}} + 3q\frac{f_6f_{12}^3}{f_2f_4} \pmod{4}. \tag{3.2.123}$$

Extracting the terms involving q^{2n} from (3.2.123) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+3)q^n \equiv 3\frac{f_2^3f_3^3}{f_1^3f_6} \pmod{4}. \tag{3.2.124}$$

Invoking (1.31) into (3.2.124), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+3)q^n \equiv 3\frac{f_1f_2f_3^3}{f_6} \pmod{4}. \tag{3.2.125}$$

Substituting (1.74) into (3.2.125), we find that

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+3)q^n \equiv 3\frac{f_3^2f_9^4}{f_{18}^2} - 3q\frac{f_3^3f_9f_{18}}{f_6} - 6q^2\frac{f_3^4f_{18}^4}{f_6^2f_9^2} \pmod{4}. \tag{3.2.126}$$

Extracting the terms involving q^{3n} from (3.2.126) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+3)q^n \equiv 3\frac{f_1^2 f_3^4}{f_6^2} \pmod{4}. \tag{3.2.127}$$

Invoking (1.31) into (3.2.127), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+3)q^n \equiv 3f_1^2 \pmod{4}. \tag{3.2.128}$$

Extracting the terms involving q^{2n} from (3.2.34) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(72n+6)q^n \equiv 3f_1^2 \pmod{4}. \tag{3.2.129}$$

In view of congruences (3.2.129) and (3.2.128), we obtain (3.2.109).

Extracting the terms involving q^{3n+2} from (3.2.126), dividing by q^2 and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2\frac{f_1^4 f_6^4}{f_2^2 f_3^2} \pmod{4}. \tag{3.2.130}$$

Invoking (1.31) into (3.2.130), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2\frac{f_6^4}{f_3^2} \pmod{4}. \tag{3.2.131}$$

Congruences (3.2.106) and (3.2.107) follow by extracting the terms involving q^{3n+1} and q^{3n+2} from (3.2.131).

Invoking (1.31) into (3.2.131), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+27)q^n \equiv 2\frac{f_{12}^2}{f_6} \pmod{4}. \tag{3.2.132}$$

Extracting the terms involving q^{6n} from (3.2.132) and replacing q^6 by q, we get (3.2.108).

Extracting the terms involving q^{3n+2} from (3.2.113), dividing by q^2 and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+5)q^n \equiv 2\frac{f_1^4 f_6^4}{f_2^4} \pmod{4}. \tag{3.2.133}$$

Invoking (1.31) into (3.2.133), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(6n+5)q^n \equiv 2\frac{f_6^4}{f_2^2} \pmod{4}. \tag{3.2.134}$$

Congruence (3.2.103) follows by extracting the terms involving q^{2n+1} from (3.2.134).

Extracting the terms involving q^{2n} from (3.2.134) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+5)q^n \equiv 2\frac{f_3^4}{f_1^2} \pmod{4}. \tag{3.2.135}$$

Substitute (1.42) and (1.46) in (3.2.135)

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+5)q^n$$
$$\equiv 2\frac{f_4^4 f_6^3 f_{16} f_{24}^2}{f_2^4 f_8 f_{12}^2 f_{48}} + 2q\frac{f_4^3 f_6^3 f_8^2 f_{48}}{f_2^4 f_{12} f_{16} f_{24}} + 2q\frac{f_6 f_{12}^2 f_{16} f_{24}^2}{f_2^2 f_8 f_{48}} + 2q^2\frac{f_6 f_8^2 f_{12}^3 f_{48}}{f_2^2 f_{16} f_{24}} \pmod{4}. \tag{3.2.136}$$

Extracting the terms involving q^{2n+1} from (3.2.136), dividing by q and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+17)q^n \equiv 2\frac{f_2^3 f_3^3 f_4^2 f_{24}}{f_1^4 f_6 f_8 f_{12}} + 2\frac{f_3 f_6^2 f_8 f_{12}^2}{f_1^2 f_4 f_{24}} \pmod{4}. \tag{3.2.137}$$

Invoking (1.31) into (3.2.137), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(24n+17)q^n \equiv 2f_2 f_3 f_{12} + 2f_2 f_3 f_{12} \pmod{4}, \tag{3.2.138}$$

which implies (3.2.105). □

3.2.7 Congruences modulo 4

Theorem 3.2.9. *For $n \geq 0$ and $\alpha \geq 0$,*

$$PD_{2,3}(648n+459) \equiv 0 \pmod{4}, \tag{3.2.139}$$

$$PD_{2,3}(8 \cdot 9^{\alpha+3}n + 51 \cdot 9^{\alpha+2}) \equiv 0 \pmod{4}. \tag{3.2.140}$$

Proof. Employing (1.33) into (3.2.108), we get

$$PD_{2,3}(216n+27)q^n \equiv 2f(q^3,q^6) + 2q\psi(q^9) \pmod{4}. \tag{3.2.141}$$

Congruence (3.2.139) follows by extracting the terms involving q^{3n+2} from (3.2.141).

Extracting the terms involving q^{3n+1} from (3.2.141), dividing by q and then replacing q^3 by q, we have

$$PD_{2,3}(648n+243)q^n \equiv 2\psi(q^3) \pmod{4}. \tag{3.2.142}$$

Extracting the terms involving q^{3n} from (3.2.142) and replacing q^3 by q, we obtain

$$PD_{2,3}(1944n+243)q^n \equiv 2\psi(q) \pmod{4}. \tag{3.2.143}$$

In view of congruences (3.2.108) and (3.2.143), we have

$$PD_{2,3}(1944n+243) \equiv PD_{2,3}(216n+27) \pmod{4}. \tag{3.2.144}$$

Utilizing (3.2.144) and by mathematical induction on α, we get

$$PD_{2,3}(24 \cdot 9^{\alpha+2}n + 3 \cdot 9^{\alpha+2}) \equiv PD_{2,3}(216n+27) \pmod{4}. \tag{3.2.145}$$

Using (3.2.139) into (3.2.145), we obtain (3.2.140). □

3.2.8 Congruences modulo 3

Theorem 3.2.10. *For $n \geq 0$ and $\alpha \geq 0$,*

$$PD_{2,3}(6n+3) \equiv 0 \pmod{3}, \tag{3.2.146}$$

$$PD_{2,3}(6n+5) \equiv 0 \pmod{3}, \tag{3.2.147}$$

$$PD_{2,3}(36n+30) \equiv 0 \pmod{3}, \tag{3.2.148}$$

$$PD_{2,3}(4\cdot 3^{\alpha+3}n + 10\cdot 3^{\alpha+2}) \equiv 0 \pmod{3}. \tag{3.2.149}$$

Proof. Substituting (1.73) into (3.2.4), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(n)q^n = \frac{f_6^2 f_9 f_{18}^2}{f_3^3 f_{12} f_{36}} + q\frac{f_6^4 f_9^4 f_{36}^2}{f_3^4 f_{12}^2 f_{18}^4} + 2q^2\frac{f_6^3 f_9 f_{36}^2}{f_3^3 f_{12}^2 f_{18}}. \tag{3.2.150}$$

Extracting the terms involving q^{3n} from (3.2.150) and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(3n)q^n = \frac{f_2^2 f_3 f_6^2}{f_1^3 f_4 f_{12}}. \tag{3.2.151}$$

Invoking (1.31) into (3.2.151), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(3n)q^n \equiv \frac{f_2^8}{f_4^4} \pmod{3}. \tag{3.2.152}$$

Extracting the terms involving q^{2n} from (3.2.152) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1^8}{f_2^4} \pmod{3}. \tag{3.2.153}$$

But

$$\frac{f_1^8}{f_2^4} = \frac{f_1^2 f_3^2}{f_2^4}. \tag{3.2.154}$$

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_1^2 f_3^2}{f_2^4} \pmod{3}. \tag{3.2.155}$$

Substituting (1.49) into (3.2.155), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(6n)q^n \equiv \frac{f_8^4 f_{12}^8}{f_2^2 f_4^4 f_6^2 f_{24}^4} + q^2 \frac{f_4^8 f_6^2 f_{24}^4}{f_2^6 f_8^4 f_{12}^4} - 2q \frac{f_4^2 f_{12}^2}{f_2^4} \pmod{3}. \tag{3.2.156}$$

Extracting the terms involving q^{2n+1} from (3.2.156), dividing by q and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \frac{f_2^2 f_6^2}{f_1^4} \pmod{3}. \tag{3.2.157}$$

Invoking (1.31) into (3.2.157), we obtain

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \frac{f_2^2 f_6^2}{f_1 f_3} \pmod{3}, \tag{3.2.158}$$

which implies that

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \psi(q)\psi(q^3) \pmod{3}. \tag{3.2.159}$$

Employing (1.33) into (3.2.159), we have

$$\sum_{n=0}^{\infty} PD_{2,3}(12n+6)q^n \equiv \psi(q^3)f(q^3,q^6) + q\psi(q^3)\psi(q^9) \pmod{3}. \tag{3.2.160}$$

Congruence (3.2.148) follows by extracting the terms involving q^{3n+2} from (3.2.160).

Extracting the terms involving q^{3n+1} from (3.2.160), dividing by q and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} PD_{2,3}(36n+18)q^n \equiv \psi(q)\psi(q^3) \pmod{3}. \tag{3.2.161}$$

In view of congruences (3.2.159) and (3.2.161), we obtain

$$PD_{2,3}(36n+18)q^n \equiv PD_{2,3}(12n+6) \pmod{3}. \tag{3.2.162}$$

Utilizing (3.2.162) and by mathematical induction on α, we get

$$PD_{2,3}(4\cdot 3^{\alpha+2}n+2\cdot 3^{\alpha+2})\equiv PD_{2,3}(12n+6) \pmod 3. \tag{3.2.163}$$

Using (3.2.148) into (3.2.163), we get (3.2.149).

Invoking (1.31) into (3.2.111), we get

$$\sum_{n=0}^{\infty} PD_{2,3}(2n+1)q^n \equiv \frac{f_6^4}{f_3^4} \pmod 3. \tag{3.2.164}$$

Congruences (3.2.146) and (3.2.147) follow by extracting the terms involving q^{3n+1} and q^{3n+2} from (3.2.164). □

3.3 Arithmetic properties of 3-regular bipartitions with designated summands

In this section, we study $PBD_3(n)$, the number of 3-regular bipartitions of n with designated summands and the generating function is given by

$$\sum_{n=0}^{\infty} PBD_3(n)q^n = \frac{f_6^4 f_9^2}{f_1^2 f_2^2 f_{18}^2}. \tag{3.3.1}$$

To be precise by a bipartition with designated summands, we mean a pair of partitions (ν_1,ν_2) where in partitions ν_1 and ν_2 are partitions with designated summands. Thus $PBD_3(4) = 35$ are

$$(4',\emptyset),\ (2'+2,\emptyset),\ (2+2',\emptyset),\ (2'+1'+1,\emptyset),\ (2'+1+1',\emptyset),\ (1'+1+1+1,\emptyset),$$
$$(1+1'+1+1,\emptyset),\ (1+1+1'+1,\emptyset),\ (1+1+1+1',\emptyset),\ (2',2'),\ (2',1'+1),$$
$$(2',1+1'),\ (1',1'+1+1),\ (1',1+1'+1),\ (1',1+1+1'),\ (1'+1,1'+1),$$
$$(1'+1,1+1'),\ (1+1',1'+1),\ (1+1',1+1'),\ (2'+1',1'),\ (1',2'+1'),\ (1'+1,2'),$$
$$(1+1',2'),\ (1'+1+1,1'),\ (1+1'+1,1'),\ (1+1+1',1'),\ (\emptyset,4'),\ (\emptyset,2'+2),$$
$$(\emptyset,2+2'),\ (\emptyset,2'+1'+1),\ (\emptyset,2'+1+1'),\ (\emptyset,1'+1+1+1),\ (\emptyset,1+1'+1+1),$$
$$(\emptyset,1+1+1'+1),\ (\emptyset,1+1+1+1').$$

3.3.1 Generating function for $PBD_3(2n)$ and $PBD_3(2n+1)$

Theorem 3.3.1. *We have* $n \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3(2n)q^n = \frac{f_3^2 f_6^6}{f_1^6 f_{18}^2} + q\frac{f_2^4 f_3^6 f_{18}^2}{f_1^8 f_6^2 f_9^2}, \tag{3.3.2}$$

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n = 2\frac{f_2^2 f_3^4 f_6^2}{f_1^7 f_9}. \tag{3.3.3}$$

Proof. Substituting (1.56) into (3.3.1), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} PBD_3(n)q^n &= \frac{f_6^4}{f_2^2 f_{18}^2}\left(\frac{f_{12}^6 f_{18}^2}{f_2^4 f_6^2 f_{36}^2} + 2q\frac{f_4^2 f_{12}^2 f_{18}}{f_2^5} + q^2\frac{f_4^4 f_6^2 f_{36}^2}{f_2^6 f_{12}^2}\right).\\ &= \frac{f_6^2 f_{12}^6}{f_2^6 f_{36}^2} + 2q\frac{f_4^2 f_6^4 f_{12}^2}{f_2^7 f_{18}} + q^2\frac{f_4^4 f_6^6 f_{36}^2}{f_2^8 f_{12}^2 f_{18}^2}.\end{aligned} \tag{3.3.4}$$

Extracting the terms involving q^{2n} and q^{2n+1} from the above equation, we obtain (3.3.2) and (3.3.3). □

3.3.2 Infinite families of congruences modulo 3

Theorem 3.3.2. *For each nonnegative integer* n *and* $\alpha \geq 0$,

$$PBD_3\left(4\times 3^{\alpha+2}n + 10\times 3^{\alpha+1}\right) \equiv 0 \pmod 3, \tag{3.3.5}$$

$$PBD_3\left(8\times 3^{\alpha+2}n + 8\times 3^{\alpha+2}\right) \equiv 0 \pmod 3, \tag{3.3.6}$$

$$PBD_3\left(2^{\alpha+3}n\right) \equiv 2^{\alpha} PBD_3(4n) \pmod 3, \tag{3.3.7}$$

$$\sum_{n=1}^{\infty} PBD_3(4n+2)q^n \equiv \psi(q)\psi(q^3) \pmod 3, \tag{3.3.8}$$

$$\sum_{n=1}^{\infty} PBD_3(8n+4)q^n \equiv 2\psi(q)\psi(q^3) \pmod 3. \tag{3.3.9}$$

Proof. Invoking (1.31) in (3.3.2), we find that

$$\sum_{n=0}^{\infty} PBD_3(2n)q^n \equiv 1 + q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod 3, \tag{3.3.10}$$

which implies,

$$\sum_{n=1}^{\infty} PBD_3(2n)q^n \equiv q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod 3. \tag{3.3.11}$$

Employing (1.44) into (3.3.11), we have

$$\sum_{n=1}^{\infty} PBD_3(2n)q^n \equiv q\frac{f_4^2 f_{12}^2}{f_2 f_6} - q^2\frac{f_2 f_{12}^6}{f_4^2 f_6^3} \pmod 3. \tag{3.3.12}$$

Extracting the terms containing q^{2n+1}, dividing throughout by q and then replacing q^2 by q from (3.3.12) and using the fact that $\psi(q) = \dfrac{f_2^2}{f_1}$, we get (3.3.8)

Substituting (1.33) into (3.3.8), we obtain

$$\sum_{n=1}^{\infty} PBD_3(4n+2)q^n \equiv f(q^3, q^6)\psi(q^3) + q\psi(q^3)\psi(q^9) \pmod 3, \tag{3.3.13}$$

which implies that

$$\sum_{n=1}^{\infty} PBD_3(12n+6)q^n \equiv \psi(q)\psi(q^3) \pmod 3. \tag{3.3.14}$$

From equations (3.3.8) and (3.3.14), we get

$$PBD_3(12n+6) \equiv PBD_3(4n+2) \pmod 3. \tag{3.3.15}$$

By using mathematical induction on α in (3.3.15), we have

$$PBD_3\left(4\times 3^{\alpha+1}n + 2\times 3^{\alpha+1}\right) \equiv PBD_3(4n+2) \pmod 3. \tag{3.3.16}$$

Extracting the terms containing q^{3n+2} from (3.3.13) we obtain

$$PBD_3(12n+10) \equiv 0 \pmod 3. \tag{3.3.17}$$

Using (3.3.17) in (3.3.16), we obtain (3.3.5).

Extracting the terms containing q^{2n} and replacing q^2 by q from (3.3.12), we get

$$\sum_{n=1}^{\infty} PBD_3(4n)q^n \equiv 2q\frac{f_1 f_6^6}{f_2^2 f_3^3} \pmod 3. \tag{3.3.18}$$

Employing (1.44) into (3.3.18), we obtain

$$\sum_{n=1}^{\infty} PBD_3(4n)q^n \equiv 2q\frac{f_4^2 f_{12}^2}{f_2 f_6} - 2q^2\frac{f_2 f_{12}^6}{f_4^4 f_6^3} \pmod 3. \tag{3.3.19}$$

Congruence (3.3.9) obtained by extracting the terms containing q^{2n+1} from (3.3.19) and using the fact that $\psi(q) = \dfrac{f_2^2}{f_1}$.

Substituting (1.33) into (3.3.9), we have

$$\sum_{n=1}^{\infty} PBD_3(8n+4)q^n \equiv 2f(q^3,q^6)\psi(q^3) + 2q\psi(q^3)\psi(q^9) \pmod 3. \tag{3.3.20}$$

Extracting the terms containing q^{3n+1} and q^{3n+2} from the above equation, we obtain

$$\sum_{n=1}^{\infty} PBD_3(24n+12)q^n \equiv 2\psi(q)\psi(q^3) \pmod 3 \tag{3.3.21}$$

and

$$PBD_3(24n+20) \equiv 0 \pmod 3. \tag{3.3.22}$$

In view of the congruences (3.3.9) and (3.3.21), we get

$$PBD_3(24n+12) \equiv PBD_3(8n+4) \pmod 3. \tag{3.3.23}$$

Utilizing (3.3.23) and by mathematical induction on α, we arrive at

$$PBD_3\left(8\times 3^{\alpha+1}n + 8\times 3^{\alpha+1}\right) \equiv PBD_3(8n+4) \pmod 3. \tag{3.3.24}$$

Using (3.3.22) in (3.3.24), we obtain (3.3.6).

Extracting the terms containing q^{2n} and replacing q^2 by q from (3.3.19) , we have

$$\sum_{n=1}^{\infty} PBD_3(8n)q^n \equiv q\frac{f_1 f_6^6}{f_2^4 f_3^3} \pmod 3. \tag{3.3.25}$$

In view of the congruences (3.3.25) and (3.3.18), we obtain

$$PBD_3(8n) \equiv 2\cdot PBD_3(4n) \pmod 3. \tag{3.3.26}$$

Utilizing (3.3.26) and by mathematical induction on α, we arrive at (3.3.7). □

Theorem 3.3.3. *Let p be a prime with $\left(\frac{-3}{p}\right) = -1$. Then for any nonnegative integers α,*

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}n + 2p^{2\alpha}\right)q^n \equiv \psi(q)\psi(q^3) \pmod 3, \tag{3.3.27}$$

and for $n \geq 0$, $1 \leq j \leq p-1$,

$$PBD_3\left(4p^{2\alpha+1}(pn+j) + 2p^{2\alpha+2}\right) \equiv 0 \pmod 3. \tag{3.3.28}$$

Proof. Equation (3.3.8) is the $\alpha = 0$ case of (3.3.27). If we assume that (3.3.27) holds for some $\alpha \geq 0$, then, substituting (1.37) in (3.3.27),

$$\begin{aligned}
&\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}n + 2p^{2\alpha}\right)q^n \\
&\equiv \left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right) \\
&\times\left(\sum_{m=0}^{\frac{p-3}{2}} q^{3\frac{m^2+m}{2}} f\left(q^{3\frac{p^2+(2m+1)p}{2}}, q^{3\frac{p^2-(2m+1)p}{2}}\right) + q^{3\frac{p^2-1}{8}}\psi(q^{3p^2})\right) \pmod 3.
\end{aligned} \tag{3.3.29}$$

For any odd prime p, and $0 \leq m_1, m_2 \leq (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2} + 3\frac{m_2^2+m_2}{2} \equiv \frac{4p^2-4}{8} \pmod p,$$

which implies that

$$(2m_1+1)^2 + 3(2m_2+1)^2 \equiv 0 \pmod p. \tag{3.3.30}$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the congruence (3.3.30) is $m_1 = m_2 = \dfrac{p-1}{2}$. Therefore, equating the coefficients of $q^{pn+\frac{4p^2-4}{8}}$ from both sides of (3.3.29), dividing throughout by $q^{\frac{4p^2-4}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha}\left(pn + \frac{4p^2-4}{8}\right) + 2p^{2\alpha}\right)q^n \equiv \psi(q^p)\psi(q^{3p}) \pmod 3. \tag{3.3.31}$$

Equating the coefficients of q^{pn} on both sides of (3.3.31) and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(4p^{2\alpha+2}n + 2p^{2\alpha+2}\right)q^n \equiv \psi(q)\psi(q^3) \pmod 3, \tag{3.3.32}$$

which is the $\alpha+1$ case of (3.3.27).

Extracting the terms involving q^{pn+j} for $1 \le j \le p-1$ in (3.3.31), we get (3.3.28). □

Theorem 3.3.4. *Let p be a prime with $\left(\frac{-3}{p}\right) = -1$. Then for any nonnegative integers α,*

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}n + 4p^{2\alpha}\right)q^n \equiv 2\psi(q)\psi(q^3) \pmod 3, \tag{3.3.33}$$

and for $n \ge 0$, $1 \le j \le p-1$,

$$PBD_3\left(8p^{2\alpha+1}(pn+j) + 4p^{2\alpha+2}\right) \equiv 0 \pmod 3. \tag{3.3.34}$$

Proof. Equation (3.3.9) is the $\alpha = 0$ case of (3.3.33). If we assume that (3.3.33) holds for some $\alpha \ge 0$, then, substituting (1.37) in (3.3.33),

$$\begin{aligned}
&\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}n + 4p^{2\alpha}\right)q^n \\
&\equiv 2\left(\sum_{m=0}^{\frac{p-3}{2}} q^{\frac{m^2+m}{2}} f\left(q^{\frac{p^2+(2m+1)p}{2}}, q^{\frac{p^2-(2m+1)p}{2}}\right) + q^{\frac{p^2-1}{8}}\psi(q^{p^2})\right) \\
&\times\left(\sum_{m=0}^{\frac{p-3}{2}} q^{3\frac{m^2+m}{2}} f\left(q^{3\frac{p^2+(2m+1)p}{2}}, q^{3\frac{p^2-(2m+1)p}{2}}\right) + q^{3\frac{p^2-1}{8}}\psi(q^{3p^2})\right) \pmod 3.
\end{aligned} \tag{3.3.35}$$

For any odd prime p, and $0 \le m_1, m_2 \le (p-3)/2$, consider the congruence

$$\frac{m_1^2+m_1}{2} + 3\frac{m_2^2+m_2}{2} \equiv \frac{4p^2-4}{8} \pmod p,$$

which implies that

$$(2m_1+1)^2 + 3(2m_2+1)^2 \equiv 0 \pmod p. \tag{3.3.36}$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the congruence (3.3.36) is $m_1 = m_2 = \dfrac{p-1}{2}$.

Therefore, equating the coefficients of $q^{pn+\frac{4p^2-4}{8}}$ from both sides of (3.3.35), dividing throughout by $q^{\frac{4p^2-4}{8}}$ and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha}\left(pn+\frac{4p^2-4}{8}\right)+4p^{2\alpha}\right)q^n \equiv 2\psi(q^p)\psi(q^{3p}) \pmod{3}. \tag{3.3.37}$$

Equating the coefficients of q^{pn} on both sides of (3.3.37) and then replacing q^p by q, we obtain

$$\sum_{n=1}^{\infty} PBD_3\left(8p^{2\alpha+2}n+4p^{2\alpha+2}\right)q^n \equiv 2\psi(q)\psi(q^3) \pmod{3}, \tag{3.3.38}$$

which is the $\alpha+1$ case of (3.3.33).

Extracting the terms involving q^{pn+j} for $1 \leq j \leq p-1$ in (3.3.37), we arrive at (3.3.34). □

3.3.3 Congruences modulo 6

Theorem 3.3.5. *For each $n \geq 0$,*

$$PBD_3(18n+15) \equiv 0 \pmod{6}, \tag{3.3.39}$$

$$\sum_{n=0}^{\infty} PBD_3(18n+3)q^n \equiv 4f_1f_3 \pmod{6}. \tag{3.3.40}$$

Proof. Invoking (1.31) in (3.3.3), we have

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2\frac{f_1^2f_2^2f_3f_6^2}{f_9} \pmod{18}. \tag{3.3.41}$$

Employing (1.74) into (3.3.41) and extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (3.3.41), we obtain

$$\sum_{n=0}^{\infty} PBD_3(6n+3)q^n \equiv 14\frac{f_2^3f_3^4}{f_6} + 8q\frac{f_1^3f_6^8}{f_3^5} \pmod{18}. \tag{3.3.42}$$

Invoking (1.31) in (3.3.42), we see that

$$\sum_{n=0}^{\infty} PBD_3(6n+3)q^n \equiv 4f_3^4 + 4q\frac{f_6^8}{f_3^4} \pmod{6}. \tag{3.3.43}$$

Congruence (3.3.39) follows by extracting the terms containing q^{3n+2} from the above equation.

Extracting the terms containing q^{3n} and replacing q^3 by q from (3.3.43), we arrive at

$$\sum_{n=0}^{\infty} PBD_3(18n+3)q^n \equiv 4f_1^4 \pmod{6}, \tag{3.3.44}$$

which implies,

$$\sum_{n=0}^{\infty} PBD_3(18n+3)q^n \equiv 4f_1 f_1^3 \pmod{6}. \tag{3.3.45}$$

Invoking (1.31) in (3.3.45) we get (3.3.40). □

Theorem 3.3.6. *If $p \geq 5$ is a prime such that $\left(\frac{-3}{p}\right) = -1$. Then for all integers $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} PBD_3\left(18p^{2\alpha}n + 3p^{2\alpha}\right)q^n \equiv 4f_1 f_3 \pmod{6}. \tag{3.3.46}$$

Proof. From (3.3.40), we have

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2} + 3 \times \frac{3m^2+m}{2} \equiv \frac{4p^2-4}{24} \pmod{p}.$$

This is equivalent to

$$(6k+1)^2 + 3(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-3}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, from Lemma (1.2.5),

$$\sum_{n=0}^{\infty} PBD_3\left(18\left(p^2 n + 4 \times \frac{p^2-1}{24}\right) + 3\right)q^n \equiv 4f_1 f_3 \pmod{6}. \tag{3.3.47}$$

Using (3.3.40), (3.3.47), and induction on α, we arrive at (3.3.46). □

Theorem 3.3.7. *Let $p \geq 5$ be prime and $\left(\frac{-3}{p}\right) = -1$. Then for all integers $n \geq 0$ and $\alpha \geq 1$,*

$$PBD_3\left(18p^{2\alpha}n + p^{2\alpha-1}(3p+18j)\right) \equiv 0 \pmod{6}, \tag{3.3.48}$$

where $j = 1, 2, \ldots, p-1$.

Proof. From Lemma (1.2.5) and Theorem (3.3.6), for each $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(18\left(p^2n + 4 \times \frac{p^2-1}{24}\right) + 3\right) q^n \equiv 4 f_1 f_3 \pmod{6}. \tag{3.3.49}$$

That is,

$$\sum_{n=0}^{\infty} PBD_3\left(18p^{2\alpha+1}n + 3p^{2\alpha+2}\right) q^n \equiv 4 f_p f_{3p} \pmod{6}. \tag{3.3.50}$$

Since there are no terms on the right of (3.3.50) where the powers of q are congruent to $1, 2, \ldots, p-1$ modulo p,

$$PBD_3\left(18p^{2\alpha+1}(pn+j) + 3p^{2\alpha+2}\right) \equiv 0 \pmod{6}, \tag{3.3.51}$$

for $j = 1, 2, \ldots, p-1$. Therefore, for $j = 1, 2, \ldots, p-1$ and $\alpha \geq 1$, we obtain (3.3.48). □

3.3.4 Congruences and Infinite families of congruences modulo 4

Theorem 3.3.8. *For each $n \geq 0$,*

$$PBD_3(12n+7) \equiv 0 \pmod 4, \tag{3.3.52}$$

$$PBD_3(12n+11) \equiv 0 \pmod 4, \tag{3.3.53}$$

$$PBD_3(24n+17) \equiv 0 \pmod 4, \tag{3.3.54}$$

$$PBD_3(36n+27) \equiv 0 \pmod 4, \tag{3.3.55}$$

$$PBD_3(72n+39) \equiv 0 \pmod 4, \tag{3.3.56}$$

$$PBD_3(72n+57) \equiv 0 \pmod 4, \tag{3.3.57}$$

$$PBD_3(216n+153) \equiv 0 \pmod 4, \tag{3.3.58}$$

$$\sum_{n=0}^{\infty} PBD_3(72n+3)q^n \equiv 2f_1 \pmod 4, \tag{3.3.59}$$

$$\sum_{n=0}^{\infty} PBD_3(72n+15)q^n \equiv 2f_1f_4 \pmod 4. \tag{3.3.60}$$

Proof. Invoking (1.31) in (3.3.3), we find that

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2\frac{f_1f_6^4}{f_2^2f_9} \pmod 8. \tag{3.3.61}$$

Employing (1.57) into (3.3.61), we obtain

$$\sum_{n=0}^{\infty} PBD_3(2n+1)q^n \equiv 2\frac{f_6^3f_{12}^3}{f_2f_4f_{18}^2} - 2q\frac{f_4f_6^5f_{36}^2}{f_2^2f_{12}f_{18}^3} \pmod 8. \tag{3.3.62}$$

Extracting the terms containing q^{2n+1}, dividing throughout by q and then replacing q^2 by q from the above equation, we get

$$\sum_{n=0}^{\infty} PBD_3(4n+3)q^n \equiv 6\frac{f_2f_3^5f_{18}^2}{f_1^2f_6f_9^3} \pmod 8. \tag{3.3.63}$$

But

$$6\frac{f_2f_3^5f_{18}^2}{f_1^2f_6f_9^3} \equiv 6\frac{f_2f_3^5f_9}{f_1^2f_6} \pmod 8. \tag{3.3.64}$$

Invoking (1.31) in (3.3.64), we get

$$\sum_{n=0}^{\infty} PBD_3(4n+3)q^n \equiv 2f_3f_6f_9 \pmod 4. \tag{3.3.65}$$

Congruences (3.3.52) and (3.3.53) follow by extracting the terms containing q^{3n+1} and q^{3n+2} from (3.3.65).

Extracting the terms containing q^{3n} and replacing q^3 by q from (3.3.65). we obtain

$$\sum_{n=0}^{\infty} PBD_3(12n+3)q^n \equiv 2f_1f_2f_3 \pmod 4. \tag{3.3.66}$$

Substituting (1.74) into (3.3.66), we find that

$$\sum_{n=0}^{\infty} PBD_3(12n+3)q^n \equiv 2\frac{f_6f_9^4}{f_{18}^2} - 2qf_3f_9f_{18} \pmod 4. \tag{3.3.67}$$

Congruence (3.3.55) obtained by extracting the terms containing q^{3n+2} from (3.3.67).

Extracting the terms containing q^{3n} and replacing q^3 by q from the above equation. we arrive at

$$\sum_{n=0}^{\infty} PBD_3(36n+3)q^n \equiv 2\frac{f_2f_3^4}{f_6^2} \pmod 4. \tag{3.3.68}$$

Using (1.31) in (3.3.68), we obtain

$$\sum_{n=0}^{\infty} PBD_3(36n+3)q^n \equiv 2f_2 \pmod 4. \tag{3.3.69}$$

Congruences (3.3.56) and (3.3.59) follow by extracting the terms containing q^{2n} and q^{2n+1} from (3.3.69).

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (3.3.67), we obtain

$$\sum_{n=0}^{\infty} PBD_3(36n+15)q^n \equiv 2f_1f_3f_6 \pmod 4. \tag{3.3.70}$$

Employing (1.49) into (3.3.70), we find that

$$\sum_{n=0}^{\infty} PBD_3(36n+15)q^n \equiv 2\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - 2q\frac{f_4^4 f_6^2 f_{24}^2}{f_2 f_8^2 f_{12}^2} \pmod{4}. \tag{3.3.71}$$

Extracting the terms containing q^{2n} and then replacing q^2 by q from (3.3.71), we obtain

$$\sum_{n=0}^{\infty} PBD_3(72n+15)q^n \equiv 2\frac{f_1 f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{4}. \tag{3.3.72}$$

Using (1.31) in (3.3.72) we arrive at (3.3.60).

Extracting the terms containing q^{2n} and replacing q^2 by q from (3.3.62), we get

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_3^3 f_6^3}{f_1 f_2 f_9^2} \pmod{8}. \tag{3.3.73}$$

Using (1.31) in (3.3.73), we have

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_3^3 f_6^3}{f_1 f_2 f_{18}} \pmod{4}. \tag{3.3.74}$$

Substituting (1.42) into (3.3.74), we arrive at

$$\sum_{n=0}^{\infty} PBD_3(4n+1)q^n \equiv 2\frac{f_4^3 f_6^5}{f_2^3 f_{12} f_{18}} + 2q\frac{f_6^3 f_{12}^3}{f_2 f_4 f_{18}} \pmod{4}. \tag{3.3.75}$$

Extracting the terms containing q^{2n} and replacing q^2 by q from (3.3.75), we obtain

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_2^3 f_3^5}{f_1^3 f_6 f_9} \pmod{4}. \tag{3.3.76}$$

But

$$\frac{f_2^3 f_3^5}{f_1^3 f_6 f_9} \equiv \frac{f_2^2 f_3 f_6}{f_1 f_9} \pmod{2}. \tag{3.3.77}$$

Which yields

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_2^2 f_3 f_6}{f_1 f_9} \pmod{4}. \tag{3.3.78}$$

Using Jacobi's triple product identity and $\psi(q) = \frac{f_2^2}{f_1}$ in (1.33), we arrive at

$$\frac{f_2^2}{f_1} = \frac{f_6 f_9^2}{f_3 f_{18}} + q\frac{f_{18}^2}{f_9}. \tag{3.3.79}$$

Employing (3.3.79) into (3.3.78), we get

$$\sum_{n=0}^{\infty} PBD_3(8n+1)q^n \equiv 2\frac{f_6^2 f_9}{f_{18}} + 2q\frac{f_3 f_6 f_{18}^2}{f_9^2} \pmod{4}. \tag{3.3.80}$$

Congruence (3.3.54) obtained by extracting the terms containing q^{3n+2} from the above equation.

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (3.3.80), we obtain

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2\frac{f_1 f_2 f_6^2}{f_3^2} \pmod{4}. \tag{3.3.81}$$

Using (1.31) in (3.3.81), we have

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2f_1 f_2 f_6 \pmod{4}. \tag{3.3.82}$$

Substituting (1.74) into (3.3.82), we obtain

$$\sum_{n=0}^{\infty} PBD_3(24n+9)q^n \equiv 2\frac{f_6^2 f_9^4}{f_3 f_{18}^2} - 2q f_6 f_9 f_{18} \pmod{4}. \tag{3.3.83}$$

Congruence (3.3.57) follows from (3.3.83) and extracting the terms containing q^{3n} and replacing q^3 by q from the above equation. we find that

$$\sum_{n=0}^{\infty} PBD_3(72n+9)q^n \equiv 2\frac{f_2^2 f_3^4}{f_1 f_6^2} \pmod{4}. \tag{3.3.84}$$

Using (1.31) in (3.3.84), we get

$$\sum_{n=0}^{\infty} PBD_3(72n+9)q^n \equiv 2\frac{f_2^2}{f_1} \equiv 2\psi(q) \pmod{4}. \tag{3.3.85}$$

Substituting (1.33) into (3.3.85) and extracting the terms containing q^{3n+2}, we arrive at (3.3.58). □

Theorem 3.3.9. *For any prime $p \geq 5$, $\alpha \geq 0$ and $n \geq 0$,*

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha}n + 3p^{3\alpha}\right)q^n \equiv 2f_1 \pmod{4}. \tag{3.3.86}$$

Proof. Employing Lemma (1.2.5) into (3.3.59), it can be see that

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(pn + \frac{p^2-1}{24}\right) + 3\right)q^n \equiv 2f_p \pmod{4}, \tag{3.3.87}$$

which implies that

$$\sum_{n=0}^{\infty} PBD_3\left(72p^2n + 3p^3\right)q^n \equiv 2f_1 \pmod{4}. \tag{3.3.88}$$

Therefore,

$$PBD_3\left(72p^2n + 3p^3\right) \equiv PBD_3(72n+3) \pmod{4}.$$

Using the above relation and by induction on α, we arrive at (3.3.86). □

Theorem 3.3.10. *For any prime $p \geq 5$, $\alpha \geq 0$, $n \geq 0$ and $l = 1, 2, ... p-1$,*

$$PBD_3\left(72p^{2\alpha}(pn+l) + 3p^{3\alpha}\right) \equiv 0 \pmod{4}. \tag{3.3.89}$$

Proof. Combining (3.3.87) with Theorem (3.3.9), we derive that for $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}n + 3p^{3\alpha}\right) \equiv 2f_p \pmod{4}.$$

Therefore, it follows that

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}(pn+l) + 3p^{3\alpha}\right) \equiv 0 \pmod{4}.$$

where $l = 1, 2, ..., p-1$, we obtain (3.3.89). □

Theorem 3.3.11. *If $p \geq 5$ is a prime such that $\left(\frac{-4}{p}\right) = -1$. Then for all integers $\alpha \geq 0$,*

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha}n + 15p^{2\alpha}\right)q^n \equiv 2f_1f_4 \pmod{4}. \tag{3.3.90}$$

Proof. From (3.3.60), we have

For a prime $p \geq 5$ and $-(p-1)/2 \leq k, m \leq (p-1)/2$, consider

$$\frac{3k^2+k}{2} + 4 \times \frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod{p}.$$

This is equivalent to

$$(6k+1)^2 + 4(6m+1)^2 \equiv 0 \pmod{p}.$$

Since $\left(\frac{-4}{p}\right) = -1$, the only solution of the above congruence is $k = m = (\pm p - 1)/6$. Therefore, from Lemma (1.2.5),

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(p^2n + 5 \times \frac{p^2-1}{24}\right) + 15\right)q^n \equiv 2f_1f_4 \pmod{4}. \tag{3.3.91}$$

Using (3.3.60), (3.3.91), and induction on α, we get (3.3.90). □

Theorem 3.3.12. *Let $p \geq 5$ be prime and $\left(\frac{-4}{p}\right) = -1$. Then for all integers $n \geq 0$ and $\alpha \geq 1$,*

$$PBD_3\left(72p^{2\alpha}n + p^{2\alpha-1}(15p + 72j)\right) \equiv 0 \pmod{4}, \tag{3.3.92}$$

where $j = 1, 2, \ldots, p-1$.

Proof. From Lemma (1.2.5) and Theorem (3.3.11), for each $\alpha \geq 0$,

$$\sum_{n=0}^{\infty} PBD_3\left(72\left(p^2n + 5 \times \frac{p^2-1}{24}\right) + 15\right)q^n \equiv 2f_1f_4 \pmod{4}. \tag{3.3.93}$$

That is,

$$\sum_{n=0}^{\infty} PBD_3\left(72p^{2\alpha+1}n + 15p^{2\alpha+2}\right)q^n \equiv 2f_pf_{4p} \pmod{4}. \tag{3.3.94}$$

Since there are no terms on the right of (3.3.94) where the powers of q are congruent to

$1, 2, \ldots, p-1$ modulo p,

$$PBD_3\left(72p^{2\alpha+1}(pn+j)+15p^{2\alpha+2}\right) \equiv 0 \pmod 4, \tag{3.3.95}$$

for $j = 1, 2, \ldots, p-1$. Therefore, for $j = 1, 2, \ldots, p-1$ and $\alpha \geq 1$, we arrive at (3.3.92). □

Chapter 4

ANDREWS' SINGULAR OVERPARTITIONS

4.1 Introduction

In the introductory chapter, we defined the definition of Andrews' singular overpartition functions denoted by $\overline{C}_{k,i}(n)$. Chen et al. [16] have proved some congruences modulo 2, 3, 4, and 8 for $\overline{C}_{3,1}(n)$. They also proved some congruence for $\overline{C}_{4,1}(n)$, $\overline{C}_{6,1}(n)$ and $\overline{C}_{6,2}(n)$ modulo powers of 2 and 3. More recently Ahmed and Baruah [1] have found some new congruences for $\overline{C}_{3,1}(n)$, $\overline{C}_{8,2}(n)$, $\overline{C}_{12,4}(n)$, $\overline{C}_{24,8}(n)$ and $\overline{C}_{48,16}(n)$ modulo 18, 36. Chen [15] has also found some congruences modulo powers of 2 for $\overline{C}_{3,1}(n)$, $\overline{C}_{4,1}(n)$. Yao [80] has proved congruences modulo 16, 32, 64 for $\overline{C}_{3,1}(n)$. Naika at el. [49] have found some congruences modulo 6, 12, 16, 18, 24, 48, and 72 for $\overline{C}_{3,1}(n)$.

4.2 Andrews' singular overpartitions with odd parts

In this section, we define the function $\overline{CO}_{\delta,i}(n)$, the number of singular overpartitions of n into odd parts such that no part is divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. For $0 < i < \delta$, the generating function of $\overline{CO}_{\delta,i}(n)$ is define by

$$\sum_{n=0}^{\infty} \overline{CO}_{\delta,i}(n) q^n = \frac{(q^{\delta};q^{2\delta})_\infty(-q^i;q^{\delta})_\infty(-q^{\delta-i};q^{\delta})_\infty}{(q;q^2)_\infty(-q^{2i};q^{2\delta})_\infty(-q^{2(\delta-i)};q^{2\delta})_\infty}. \tag{4.2.1}$$

Reference [56], [57] and [54] is based on this chapter

4.2.1 Congruences modulo 8 and 16

Theorem 4.2.1. *For each integer $n \geq 0$,*

$$\overline{CO}_{3,1}(12n+7) \equiv 0 \pmod{8}, \tag{4.2.2}$$

$$\overline{CO}_{3,1}(24n+19) \equiv 0 \pmod{16}, \tag{4.2.3}$$

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+7)q^n \equiv \psi(q)f_4 \pmod{16}. \tag{4.2.4}$$

Proof. Setting $\delta = 3$ and $i = 1$ in (4.2.1), we find that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(n)q^n = \frac{(q^2;q^2)_\infty^3 (q^3;q^3)_\infty^2 (q^{12};q^{12})}{(q^6;q^6)_\infty^3 (q^4;q^4)_\infty (q;q)_\infty^2}. \tag{4.2.5}$$

Substituting (1.47) into (4.2.5), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(n)q^n = \frac{f_4^3 f_{12}^3}{f_2^2 f_6^2 f_8 f_{24}} + 2q\frac{f_8 f_{24}}{f_2 f_6}, \tag{4.2.6}$$

which yields, for each $n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n = 2\frac{f_4 f_{12}}{f_1 f_3}. \tag{4.2.7}$$

Employing (1.73) into (4.2.7), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n = 2\frac{f_{12}^2 f_{18}^4}{f_3^4 f_{36}^2} + 2q\frac{f_6^2 f_9^3 f_{12} f_{36}}{f_3^5 f_{18}^2} + 4q^2\frac{f_6 f_{12} f_{18} f_{36}}{f_3^4}. \tag{4.2.8}$$

Extracting the terms involving q^{3n} in the above equation and replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+1)q^n = 2\frac{f_4^2 f_6^4}{f_1^4 f_{12}^2}. \tag{4.2.9}$$

Using (1.31) in (4.2.9), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+1)q^n \equiv 2f_2^2 \pmod{8}. \tag{4.2.10}$$

Congruence (4.2.2) follows by extracting the terms involving q^{2n+1} from (4.2.10).

Collecting the terms involving q^{2n} from (4.2.10) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+1) \equiv 2f_1^2 \pmod{8}. \tag{4.2.11}$$

Substituting (1.41) into (4.2.9), we find that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+1)q^n = 2\frac{f_4^{16}f_6^4}{f_2^{14}f_8^4f_{12}^2} + 8q\frac{f_4^4f_6^4f_8^4}{f_2^{10}f_{12}^2}, \tag{4.2.12}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+7)q^n = 8\frac{f_2^4f_3^4f_4^4}{f_1^{10}f_6^2}. \tag{4.2.13}$$

Using (1.31) in (4.2.13), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+7)q^n \equiv 8f_2^7 \pmod{16}. \tag{4.2.14}$$

Extracting the terms involving q^{2n+1} from (4.2.14) we get (4.2.3).

Collecting the terms involving q^{2n} from (4.2.14) and replacing q^2 by q, reduces to

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+7)q^n \equiv 8f_1^7 \pmod{16}, \tag{4.2.15}$$

Using (1.31) in (4.2.15), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+7)q^n \equiv 8\left(\frac{f_2^2}{f_1}\right)f_4 \pmod{16}. \tag{4.2.16}$$

Using (1.37) in (4.2.16), we arrive at (4.2.4). □

Theorem 4.2.2. *For any prime $p \equiv 5 \pmod{6}$, $\alpha \geq 1$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2p^{2\alpha}n + p^{2\alpha})q^n \equiv 2\psi(q)\psi(q^3) \pmod{4}. \tag{4.2.17}$$

Proof. Using (1.31) in (4.2.7), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n \equiv 2\frac{f_2^2 f_6^2}{f_1 f_3} \pmod 4. \tag{4.2.18}$$

Using (1.37) in (4.2.18), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n \equiv 2\psi(q)\psi(q^3) \pmod 4. \tag{4.2.19}$$

Define

$$\sum_{n=0}^{\infty} g(n)q^n = \psi(q)\psi(q^3). \tag{4.2.20}$$

Combining (4.2.19) and (4.2.20), we find that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n \equiv 2\sum_{n=0}^{\infty} g(n)q^n \pmod 4. \tag{4.2.21}$$

Now, we consider the congruence equation

$$\frac{k^2+k}{2} + 3\cdot\frac{m^2+m}{2} \equiv \frac{4p^2-4}{8} \pmod p, \tag{4.2.22}$$

which is equivalent to

$$(2k+1)^2 + 3\cdot(2m+1)^2 \equiv 0 \pmod p,$$

where $0 \le k, m \le \frac{p-1}{2}$ and p is a prime such that $(\frac{-3}{p}) = -1$. Since $(\frac{-3}{p}) = -1$ for $p \equiv 5 \pmod 6$, the congruence relation (4.2.22) holds if and only if both $k = m = \frac{p-1}{2}$. Therefore, if we substitute (1.37) into (4.2.20) and then extracting the terms in which the powers of q are congruent to $\frac{p^2-1}{2}$ modulo p and then divide by $q^{\frac{p^2-1}{2}}$, we find that

$$\sum_{n=0}^{\infty} g\left(pn + \frac{p^2-1}{2}\right)q^{pn} = \psi(q^{p^2})\psi(q^{3p^2}),$$

which implies that

$$\sum_{n=0}^{\infty} g\left(p^2 n + \frac{p^2-1}{2}\right)q^{n} = \psi(q)\psi(q^3) \tag{4.2.23}$$

and for $n \geq 0$,

$$g\left(p^2 n + pi + \frac{p^2-1}{2}\right) = 0, \tag{4.2.24}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$g\left(p^{2\alpha} n + \frac{p^{2\alpha}-1}{2}\right) = g(n). \tag{4.2.25}$$

Replacing n by $p^{2\alpha} n + \frac{p^{2\alpha}-1}{2}$ in (4.2.21), we arrive at (4.2.17). □

Theorem 4.2.3. *For any prime $p \equiv 5 \pmod 6$, $\alpha \geq 1$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24p^{2\alpha} n + 7p^{2\alpha}) q^n \equiv (-1)^{\alpha \cdot \frac{\pm p-1}{6}} \psi(q) f_4 \pmod{16}. \tag{4.2.26}$$

Proof. Define

$$\sum_{n=0}^{\infty} a(n) q^n = \psi(q) f_4. \tag{4.2.27}$$

Combining (4.2.4) and (4.2.27), we see that

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+7) q^n \equiv \sum_{n=0}^{\infty} a(n) q^n \pmod{16}. \tag{4.2.28}$$

Now, we consider the congruence equation

$$\frac{k^2+k}{2} + 4 \cdot \frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod p, \tag{4.2.29}$$

which is equivalent to

$$3 \cdot (2k+1)^2 + (12m+2)^2 \equiv 0 \pmod p,$$

where $\frac{-(p-1)}{2} \leq m \leq \frac{p-1}{2}$, $0 \leq k \leq \frac{p-1}{2}$ and p is a prime such that $\left(\frac{-3}{p}\right) = -1$. Since $\left(\frac{-3}{p}\right) = -1$ for $p \equiv 5 \pmod 6$, the congruence relation (4.2.29) holds if and only if $m = \frac{\pm p-1}{6}$ and $k = \frac{p-1}{2}$. Therefore, if we substitute (1.37) and (1.36) into (4.2.27) and then extracting the terms in which the powers of q are $pn + \frac{7p^2-7}{24}$, we arrive at

$$\sum_{n=0}^{\infty} a\left(pn + \frac{7p^2-7}{24}\right) q^{pn+\frac{7p^2-7}{24}} = (-1)^{\frac{\pm p-1}{6}} q^{\frac{7p^2-7}{24}} \psi(q^{p^2}) f_{4p^2}. \tag{4.2.30}$$

Dividing by $q^{\frac{7p^2-7}{24}}$ on both sides of (4.2.30) and on simplification, we find that

$$\sum_{n=0}^{\infty} a\left(pn + \frac{7p^2-7}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} \psi(q^p) f_{4p},$$

which implies that

$$\sum_{n=0}^{\infty} a\left(p^2 n + \frac{7p^2-7}{24}\right) q^n = (-1)^{\frac{\pm p-1}{6}} \psi(q) f_4 \tag{4.2.31}$$

and for $n \geq 0$,

$$a\left(p^2 n + pi + \frac{7p^2-7}{24}\right) = 0, \tag{4.2.32}$$

where i is an integer and $1 \leq i \leq p-1$. Combining (4.2.27) and (4.2.31), we see that for $n \geq 0$,

$$a\left(p^2 n + \frac{7p^2-7}{24}\right) = (-1)^{\frac{\pm p-1}{6}} a(n). \tag{4.2.33}$$

By (4.2.33) and mathematical induction, we deduce that for $n \geq 0$ and $\alpha \geq 0$,

$$a\left(p^{2\alpha} n + \frac{7p^{2\alpha}-7}{24}\right) = (-1)^{\alpha \cdot \frac{\pm p-1}{6}} a(n). \tag{4.2.34}$$

Replacing n by $p^{2\alpha} n + \frac{7p^{2\alpha}-7}{24}$ in (4.2.28), we arrive at (4.2.26). □

4.2.2 Congruences modulo 8

Theorem 4.2.4. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{CO}_{3,1}(36n+21) \equiv 0 \pmod{8}, \tag{4.2.35}$$

$$\overline{CO}_{3,1}(36n+3) \equiv \overline{CO}_{3,1}(12n+1) \pmod{8}, \tag{4.2.36}$$

$$\overline{CO}_{3,1}(4 \cdot 3^{\alpha+3} n + 7 \cdot 3^{\alpha+2}) \equiv 0 \pmod{8}, \tag{4.2.37}$$

$$\overline{CO}_{3,1}(36n+33) \equiv 0 \pmod{8}, \tag{4.2.38}$$

$$\overline{CO}_{3,1}(18n+15) \equiv \overline{CO}_{3,1}(6n+5) \pmod{8}. \tag{4.2.39}$$

Proof. Equating the coefficients of q^{3n+1} from both sides of (4.2.8), dividing by q and then replacing q^3 by q, we arrive at

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+3)q^n = 2\frac{f_2^2 f_3^3 f_4 f_{12}}{f_1^5 f_6^2}. \tag{4.2.40}$$

Using (1.31) in (4.2.40), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+3)q^n \equiv 2\frac{f_4 f_{12}}{f_1 f_3} \pmod{8}. \tag{4.2.41}$$

Substituting (1.73) into (4.2.41), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+3)q^n \equiv 2\frac{f_{12}^2 f_{18}^4}{f_3^4 f_{36}^2} + 2q\frac{f_6^2 f_9^3 f_{12} f_{36}}{f_3^5 f_{18}^2} + 4q^2\frac{f_6 f_{12} f_{18} f_{36}}{f_3^4} \pmod{8}, \tag{4.2.42}$$

which implies that for all $n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+3)q^n \equiv 2\frac{f_4^2 f_6^4}{f_1^4 f_{12}^2} \pmod{8}. \tag{4.2.43}$$

Using (1.31) in (4.2.43), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+3)q^n \equiv 2f_2^2 \pmod{8}. \tag{4.2.44}$$

Equating the coefficients of q^{2n+1} from both sides of (4.2.44), dividing by q and then replacing q^2 by q, we arrive at (4.2.35).

From (4.2.44), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+3)q^n \equiv 2f_1^2 \pmod{8}. \tag{4.2.45}$$

In view of congruences (4.2.45) and (4.2.11), we obtain (4.2.36).

Extracting the terms involving q^{3n+1} from (4.2.42), dividing by q and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+9)q^n \equiv 2\frac{f_2^2 f_3^3 f_4 f_{12}}{f_1^5 f_6^2} \pmod{8}. \tag{4.2.46}$$

Using (1.31) in (4.2.46), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+9)q^n \equiv 2\frac{f_4 f_{12}}{f_1 f_3} \pmod{8}. \tag{4.2.47}$$

In view of congruences (4.2.47) and (4.2.41), we get

$$\overline{CO}_{3,1}(18n+9) \equiv \overline{CO}_{3,1}(6n+3) \pmod{8}. \tag{4.2.48}$$

Utilizing (4.2.48) and by mathematical induction on α, we arrive at

$$\overline{CO}_{3,1}(2\cdot 3^{\alpha+2}n+3^{\alpha+2}) \equiv \overline{CO}_{3,1}(6n+3) \pmod{8}. \tag{4.2.49}$$

Using (4.2.35) in (4.2.49), we obtain (4.2.37).

From (4.2.42), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+15)q^n \equiv 4\frac{f_2 f_4 f_6 f_{12}}{f_1^4} \pmod{8}. \tag{4.2.50}$$

Using (1.31) in (4.2.50), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(18n+15)q^n \equiv 4 f_2 f_6 f_{12} \pmod{8}. \tag{4.2.51}$$

Congruence (4.2.38) follows by extracting the terms involving q^{2n+1} from (4.2.51).

Extracting the terms involving q^{3n+2} from (4.2.8), dividing by q^2 and then replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+5)q^n = 4\frac{f_2 f_4 f_6 f_{12}}{f_1^4}. \tag{4.2.52}$$

Using (1.31) in (4.2.52), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(6n+5)q^n \equiv 4 f_2 f_6 f_{12} \pmod{8}. \tag{4.2.53}$$

Combining (4.2.51) and (4.2.53), we arrive at (4.2.39). □

Theorem 4.2.5. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{CO}_{3,1}(12n+7) \equiv 0 \pmod{8}, \tag{4.2.54}$$

$$\overline{CO}_{3,1}(12n+11) \equiv 0 \pmod{8}, \tag{4.2.55}$$

$$\overline{CO}_{3,1}(108n+63) \equiv 0 \pmod{8}, \tag{4.2.56}$$

$$\overline{CO}_{3,1}(108n+99) \equiv 0 \pmod{8}, \tag{4.2.57}$$

$$\overline{CO}_{3,1}(972n+567) \equiv 0 \pmod{8}, \tag{4.2.58}$$

$$\overline{CO}_{3,1}(972n+891) \equiv 0 \pmod{8}, \tag{4.2.59}$$

$$\overline{CO}_{3,1}(12 \cdot 9^{\alpha+2}n + 3 \cdot 9^{\alpha+2}) \equiv \overline{CO}_{3,1}(108n+27) \pmod{8}. \tag{4.2.60}$$

Proof. Substituting (1.50) into (4.2.7), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(2n+1)q^n = 2\frac{f_8^2 f_{12}^6}{f_2^2 f_6^4 f_{24}^2} + 2q\frac{f_4^6 f_{24}^2}{f_2^4 f_6^2 f_8^2}, \tag{4.2.61}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(4n+3)q^n = 2\frac{f_2^6 f_{12}^2}{f_1^4 f_3^2 f_4^2}. \tag{4.2.62}$$

Using (1.31) in (4.2.62), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(4n+3)q^n \equiv 2\frac{f_{12}^2}{f_3^2} \pmod{8}. \tag{4.2.63}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} from (4.2.63) we get (4.2.54) and (4.2.55).

Extracting the terms involving q^{3n} from (4.2.63) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+3)q^n \equiv 2\frac{f_4^2}{f_1^2} \pmod{8}. \tag{4.2.64}$$

Substituting (1.73) into (4.2.64) and equating the terms q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+27)q^n \equiv 2\frac{f_2^4 f_3^6 f_{12}^2}{f_1^8 f_6^4} \pmod{8}. \tag{4.2.65}$$

Using (1.31) in (4.2.65), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+27)q^n \equiv 2f_3^6 \pmod{8}. \tag{4.2.66}$$

Congruences (4.2.56) and (4.2.57) follow by extracting the terms involving q^{3n+1} and q^{3n+2} from (4.2.65).

Extracting the terms involving q^{3n} from (4.2.66) and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(108n+27)q^n \equiv 2f_1^6 \pmod{8}, \tag{4.2.67}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(108n+27)q^n \equiv 2f_1^2 f_2^2 \pmod{8}. \tag{4.2.68}$$

Employing (1.74) into (4.2.68) and equating the terms involving q^{3n+2}, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(324n+243)q^n \equiv 2f_3^2 f_6^2 \pmod{8}. \tag{4.2.69}$$

Using (1.31) in (4.2.69), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(324n+243)q^n \equiv 2f_3^6 \pmod{8}. \tag{4.2.70}$$

Extracting the terms involving q^{3n+1} and $q^{3n+2}$from (4.2.70), we arrive at (4.2.58) and (4.2.59).

Extracting the terms involving q^{3n} from (4.2.70) and replacing q^3 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(972n+243)q^n \equiv 2f_1^6 \pmod{8}. \tag{4.2.71}$$

In view of congruences (4.2.71) and (4.2.67), we get

$$\overline{CO}_{3,1}(972n+243) \equiv \overline{CO}_{3,1}(108n+27) \pmod{8}. \tag{4.2.72}$$

Utilizing (4.2.72) and by mathematical induction on α, we arrive at (4.2.60). □

Theorem 4.2.6. *For all* $n \geq 0$ *and* $\alpha \geq 0$,

$$\overline{CO}_{3,1}(24n+14) \equiv 0 \pmod 8, \tag{4.2.73}$$

$$\overline{CO}_{3,1}(4\cdot 3^{\alpha+2}n+2\cdot 3^{\alpha+2}) \equiv 3^{\alpha+1}\overline{CO}_{3,1}(12n+6) \pmod 8, \tag{4.2.74}$$

$$\overline{CO}_{3,1}(108n+27) \equiv 3\overline{CO}_{3,1}(24n+6) \pmod 8, \tag{4.2.75}$$

$$\overline{CO}_{3,1}(72n+6) \equiv 3\overline{CO}_{3,1}(24n+2) \pmod 8, \tag{4.2.76}$$

$$\overline{CO}_{3,1}(72n+42) \equiv 0 \pmod 8, \tag{4.2.77}$$

$$\overline{CO}_{3,1}(72n+66) \equiv 0 \pmod 8, \tag{4.2.78}$$

$$\overline{CO}_{3,1}(24n+22) \equiv 0 \pmod 8, \tag{4.2.79}$$

$$\overline{CO}_{3,1}(36n+30) \equiv \overline{CO}_{3,1}(12n+10) \pmod 8. \tag{4.2.80}$$

Proof. From (4.2.6), we have

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(2n)q^n = \frac{f_2^3 f_6^3}{f_1^2 f_3^2 f_4 f_{12}}. \tag{4.2.81}$$

Substituting (1.50) into (4.2.81) and equating the terms q^{2n+1}, we get

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(4n+2)q^n = 2\frac{f_2^3 f_6^3}{f_1^3 f_3^3}. \tag{4.2.82}$$

Using (1.31) in (4.2.82), we obtain

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(4n+2)q^n \equiv 2\frac{f_6^3}{f_3^3}(f_1 f_2) \pmod 8. \tag{4.2.83}$$

Employing (1.74) into (4.2.83), we have

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(4n+2)q^n \equiv 2\frac{f_6^4 f_9^4}{f_3^4 f_{18}^2} - 2q\frac{f_6^3 f_9 f_{18}}{f_3^3} - 4q^2\frac{f_6^2 f_{18}^4}{f_3^2 f_9^2} \pmod 8, \tag{4.2.84}$$

which implies that

$$\sum_{n=0}^{\infty}\overline{CO}_{3,1}(12n+2)q^n \equiv 2\frac{f_2^4 f_3^4}{f_1^4 f_6^2} \pmod 8. \tag{4.2.85}$$

Using (1.31) in (4.2.85), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+2)q^n \equiv 2f_2^2 \pmod{8}. \tag{4.2.86}$$

Congruence (4.2.73) follows by extracting the terms involving q^{2n+1} from (4.2.86).

Extracting the terms involving q^{2n} from (4.2.86), we arrive at

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+2) \equiv 2f_1^2 \pmod{8}. \tag{4.2.87}$$

Extracting the terms involving q^{3n+1} from (4.2.84), dividing by q and then replacing q^{3n} by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+6)q^n \equiv 6\frac{f_2^3 f_3 f_6}{f_1^3} \pmod{8}. \tag{4.2.88}$$

Using (1.31) in (4.2.88), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+6)q^n \equiv 6(f_1 f_2) f_3 f_6 \pmod{8}. \tag{4.2.89}$$

Substituting (1.74) into (4.2.89), we arrive at

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+6)q^n \equiv 6\frac{f_6^2 f_9^4}{f_{18}^2} - 6q f_3 f_6 f_9 f_{18} - 12q^2\frac{f_3^2 f_{18}^4}{f_9^2} \pmod{8}, \tag{4.2.90}$$

which implies that for all $n \geq 0$

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+18)q^n \equiv 2f_1 f_2 f_3 f_6 \pmod{8}. \tag{4.2.91}$$

In the view of congruences (4.2.91) and (4.2.89), we have

$$\overline{CO}_{3,1}(36n+18) \equiv 3\overline{CO}_{3,1}(12n+6) \pmod{8}. \tag{4.2.92}$$

Utilizing (4.2.92) and by mathematical induction on α, we arrive at (4.2.74).

Employing (1.49) into (4.2.89), we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+6)q^n \equiv 6\frac{f_2^2 f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - 6q\frac{f_4^4 f_6^2 f_{24}^2}{f_8^2 f_{12}^2} \pmod 8. \tag{4.2.93}$$

Extracting the terms involving q^{2n} from (4.2.93) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+6)q^n \equiv 6\frac{f_1^2 f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod 8. \tag{4.2.94}$$

Using (1.31) in (4.2.94), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+6)q^n \equiv 6f_1^2 f_2^2 \pmod 8. \tag{4.2.95}$$

Combining (4.2.95) and (4.2.68), we obtain (4.2.75).

Extracting the terms involving q^{3n} from (4.2.90) and then replacing q^3 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+6)q^n \equiv 6\frac{f_2^2 f_3^4}{f_6^2} \pmod 8. \tag{4.2.96}$$

Using (1.31) in (4.2.96), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+6)q^n \equiv 6f_2^2 \pmod 8. \tag{4.2.97}$$

Congruence (4.2.77) follows by extracting the terms involving q^{2n+1} from (4.2.97).

Extracting the terms involving q^{2n} from (4.2.97) and then replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(72n+6)q^n \equiv 6f_1^2 \pmod 8. \tag{4.2.98}$$

Combining the equations (4.2.98) and (4.2.87), we arrive at (4.2.76).

Equating the coefficients of q^{3n+2} from both sides of (4.2.90), dividing by q^2 and then replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+30)q^n \equiv 4\frac{f_1^2 f_6^4}{f_3^2} \pmod 8. \tag{4.2.99}$$

Using (1.31) in (4.2.99), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(36n+30)q^n \equiv 4f_2f_6^3 \pmod{8}. \tag{4.2.100}$$

Extracting the terms involving q^{2n+1} from (4.2.100), we arrive at (4.2.78).

Equating the coefficients of q^{3n+2} from both sides of (4.2.84), dividing by q^2 and then replacing q^3 by q,

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(12n+10)q^n \equiv 4\frac{f_2^2f_6^4}{f_1^2f_3^2} \pmod{8}. \tag{4.2.101}$$

Using (1.31) in (4.2.101), we have

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(24n+22)q^n \equiv 4f_2f_6^3 \pmod{8}. \tag{4.2.102}$$

Congruence (4.2.79) follows by extracting the terms involving q^{2n+1} from (4.2.102).

In the view of congruences (4.2.102) and (4.2.100), we get (4.2.80). □

4.2.3 Congruences modulo 6

Theorem 4.2.7. *For all integers $n \geq 0$,*

$$\overline{CO}_{3,1}(12n+6) \equiv 0 \pmod{6}, \tag{4.2.103}$$
$$\overline{CO}_{3,1}(12n+10) \equiv 0 \pmod{6}. \tag{4.2.104}$$

Proof. Using (1.31) in (4.2.82), we obtain

$$\sum_{n=0}^{\infty} \overline{CO}_{3,1}(4n+2)q^n \equiv 2\frac{f_6^4}{f_3^4}. \tag{4.2.105}$$

Extracting the terms involving q^{3n+1} and q^{3n+2} from (4.2.105), we arrive at (4.2.103) and (4.2.104). □

4.3 Congruences for Andrews' singular overpartitions without multiples of k

In this section, we define the function $\overline{A}^k_{\delta,i}(n)$, the number of singular overpartitions of n without multiples of k in which no part divisible by δ and only parts $\equiv \pm i \pmod{\delta}$ may be overlined. For $0 < i < \delta$, the generating function of $\overline{A}^k_{\delta,i}(n)$ is

$$\sum_{n=0}^{\infty} \overline{A}^k_{\delta,i}(n) q^n = \frac{f(q^i;q^{\delta-i})(q^k;q^k)_\infty}{f(q^{ki};q^{k(\delta-i)})(q;q)_\infty}. \tag{4.3.1}$$

4.3.1 Congruences modulo 2^2 for $\overline{A}^3_{4,1}(n)$

Theorem 4.3.1. *For each integer $n \geq 0$,*

$$\overline{A}^3_{4,1}(4n+3) \equiv 0 \pmod{2^2}, \tag{4.3.2}$$

$$\sum_{n=0}^{\infty} \overline{A}^3_{4,1}(4n+1) q^n \equiv 2 f_2 f_3 \pmod{2^2}. \tag{4.3.3}$$

Proof. Setting $\delta = 4$, $i = 1$ and $k = 3$ in (4.3.1), we find that

$$\sum_{n=0}^{\infty} \overline{A}^3_{4,1}(n) q^n = \frac{(q^2;q^2)^2_\infty (q^3;q^3)^2_\infty}{(q;q)^2_\infty (q^6;q^6)^2_\infty}. \tag{4.3.4}$$

Substituting (1.47) into (4.3.4), we obtain

$$\sum_{n=0}^{\infty} \overline{A}^3_{4,1}(n) q^n = \frac{f_4^4 f_{12}^2}{f_2^3 f_6 f_8 f_{24}} + 2q \frac{f_4 f_8 f_{24}}{f_2^2 f_{12}}, \tag{4.3.5}$$

which yields, for each $n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{A}^3_{4,1}(2n+1) q^n = 2 \frac{f_2 f_4 f_{12}}{f_1^2 f_6}. \tag{4.3.6}$$

Using (1.31) in (4.3.6), we find that

$$\sum_{n=0}^{\infty} \overline{A}^3_{4,1}(2n+1) q^n \equiv 2 f_4 f_6 \pmod{2^2}. \tag{4.3.7}$$

Congruence (4.3.2) follows by extracting the terms involving q^{2n+1} from (4.3.7).

Collecting the terms involving q^{2n} from (4.3.7) and replacing q^2 by q, we get (4.3.3). □

Theorem 4.3.2. *For any prime $p \geq 5$ with $(\frac{-6}{p}) = -1$, $\alpha \geq 1$ and $n \geq 0$,*

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{3}\left(4 \cdot p^{2\alpha} n + \frac{5 \cdot p^{2\alpha}+1}{6}\right) q^n \equiv 2 f_2 f_3 \pmod{2^2}. \tag{4.3.8}$$

Proof. Define

$$\sum_{n=0}^{\infty} f(n) q^n = f_2 f_3 \pmod{2^2}. \tag{4.3.9}$$

Combining (4.3.3) and (4.3.9), we find that

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{3}(4n+1) q^n \equiv 2 \sum_{n=0}^{\infty} f(n) q^n \pmod{2^2}. \tag{4.3.10}$$

Now, we consider the congruence equation

$$2 \cdot \frac{3k^2+k}{2} + 3 \cdot \frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod{p}, \tag{4.3.11}$$

which is equivalent to

$$(2 \cdot (6k+1))^2 + 6 \cdot (6m+1)^2 \equiv 0 \pmod{p}.$$

where $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$ and p is a prime such that $(\frac{-6}{p}) = -1$. Since $(\frac{-6}{p}) = -1$ for $p \geq 5$, the congruence relation (4.3.11) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute (1.36) into (4.3.9) and then extracting the terms in which the powers of q are congruent to $5 \cdot \frac{p^2-1}{24}$ modulo p and then divide by $q^{5 \cdot \frac{p^2-1}{24}}$, we find that

$$\sum_{n=0}^{\infty} f\left(pn + 5 \cdot \frac{p^2-1}{24}\right) q^{pn} = f_{2p} f_{3p}, \tag{4.3.12}$$

which implies that

$$\sum_{n=0}^{\infty} f\left(p^2 n + 5 \cdot \frac{p^2-1}{24}\right) q^n = f_2 f_3 \tag{4.3.13}$$

and for $n \geq 0$,

$$f\left(p^2n + pi + 5 \cdot \frac{p^2-1}{24}\right) = 0, \tag{4.3.14}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha}n + 5 \cdot \frac{p^{2\alpha}-1}{24}\right) = f(n). \tag{4.3.15}$$

Replacing n by $p^{2\alpha}n + 5 \cdot \frac{p^{2\alpha}-1}{24}$ in (4.3.10), we arrive at (4.3.8). □

Theorem 4.3.3. *For any prime $p \geq 5$ with $\left(\frac{-6}{p}\right) = -1$, $\alpha \geq 1$ and $n \geq 0$,*

$$\overline{A}_{4,1}^{3}\left(4 \cdot p^{2\alpha+2}n + 4 \cdot p^{2\alpha+1}i + \frac{5 \cdot p^{2\alpha+2}+1}{6}\right) \equiv 0 \pmod{2^2}.$$

where $i = 1, 2, \ldots, p-1$.

Proof. Replacing n by $p^2n + pi + \frac{5 \cdot p^2 - 5}{24}$ in (4.3.15) and using (4.3.14), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{5 \cdot p^{2\alpha+2}-5}{24}\right) = 0. \tag{4.3.16}$$

Comparing the coefficients of q^n from the both sides of (4.3.10), we see that for $n \geq 0$,

$$\overline{A}_{4,1}^{3}(4n+1) \equiv 2f(n) \pmod{2^2}. \tag{4.3.17}$$

The result follows from (4.3.16) and (4.3.17). □

4.3.2 Infinite families of congruences modulo 2^2 and 2^3 for $\overline{A}_{4,1}^{5}(n)$

Theorem 4.3.4. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{A}_{4,1}^{5}\left(2^{2\alpha+5}n + \frac{7 \cdot 2^{2\alpha+3}+1}{3}\right) \equiv 0 \pmod{2^2}, \tag{4.3.18}$$

$$\overline{A}_{4,1}^{5}(8n+5) \equiv 0 \pmod{2^2}, \tag{4.3.19}$$

$$\overline{A}_{4,1}^{5}(16n+9) \equiv 0 \pmod{2^2}, \tag{4.3.20}$$

$$\overline{A}_{4,1}^{5}(8(4n+i)+7) \equiv 0 \pmod{2^2}, \tag{4.3.21}$$

where i =1, 2, 3.

$$\overline{A}_{4,1}^{5}(32(5n+i)+15) \equiv 0 \pmod{2^2}, \tag{4.3.22}$$

where i =1, 2, 3, 4.

$$\overline{A}_{4,1}^{5}(160n+7) \equiv \overline{A}_{4,1}^{5}(16n+1) \pmod{2^2}. \tag{4.3.23}$$

Proof. Setting $\delta = 4$, $i = 1$ and $k = 5$ in (4.3.1), we find that

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(n)q^n = \frac{(q^2;q^2)_\infty^2 (q^5;q^5)_\infty^2}{(q;q)_\infty^2 (q^{10};q^{10})_\infty^2}. \tag{4.3.24}$$

Employing (1.52) into (4.3.24), we get

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(n)q^n = \frac{f_8^2 f_{20}^4}{f_2^2 f_{10}^2 f_{40}^2} + q^2 \frac{f_4^6 f_{40}^2}{f_2^4 f_8^2 f_{20}^2} + 2q \frac{f_4^3 f_{20}}{f_2^3 f_{10}}. \tag{4.3.25}$$

Extracting the terms involving q^{2n+1} from (4.3.25), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(2n+1)q^n = 2\frac{f_2^3 f_{10}}{f_1^3 f_5}. \tag{4.3.26}$$

Using (1.31) in (4.3.26), we find that

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(2n+1)q^n \equiv 2f_1^3 f_5 \pmod{2^2}. \tag{4.3.27}$$

Employing (1.59) into (4.3.27), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(2n+1)q^n \equiv 2\frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2qf_2 f_{10}^3 \pmod{2^2}, \tag{4.3.28}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(4n+3)q^n \equiv 2f_1 f_5^3 \pmod{2^2}. \tag{4.3.29}$$

Substituting (1.58) into (4.3.29), we get

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(4n+3)q^n \equiv 2f_2^3 f_{10} + 2q\frac{f_2^2 f_{10}^2 f_{20}}{f_4} \pmod{2^2}. \tag{4.3.30}$$

Extracting the terms involving q^{2n} from (4.3.30) and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(8n+3)q^n \equiv 2f_1^3 f_5 \pmod{2^2}. \tag{4.3.31}$$

In view of congruences (4.3.27) and (4.3.31), we arrive at

$$\overline{A}_{4,1}^{5}(8n+3) \equiv \overline{A}_{4,1}^{5}(2n+1) \pmod{2^2}. \tag{4.3.32}$$

Utilizing (4.3.32) and by mathematical induction on α, we get

$$\overline{A}_{4,1}^{5}\left(2\cdot 4^{\alpha+1}n + \frac{2\cdot 4^{\alpha+1}+1}{3}\right) \equiv \overline{A}_{4,1}^{5}(2n+1) \pmod{2^2}. \tag{4.3.33}$$

Utilizing (4.3.19) and (4.3.33), we obtain (4.3.18).

Collecting the terms involving q^{2n} from (4.3.28) and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(4n+1)q^n \equiv 2\frac{f_1^2 f_2 f_5^2}{f_{10}} \pmod{2^2}. \tag{4.3.34}$$

Using (1.31) in (4.3.34), we find that

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(4n+1)q^n \equiv 2f_4 \pmod{2^2}, \tag{4.3.35}$$

Congruences (4.3.19) and (4.3.20) follow by extracting the terms involving q^{2n+1} and q^{4n+2} from both sides of (4.3.35).

Collecting the terms involving q^{4n} from (4.3.35) and replacing q^4 by q, we have

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(16n+1)q^n \equiv 2f_1 \pmod{2^2}, \tag{4.3.36}$$

Extracting the terms involving q^{2n+1} from (4.3.30), dividing by q and then replacing q^2

by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(8n+7)q^n \equiv 2\frac{f_1^2 f_5^2 f_{10}}{f_2} \pmod{2^2}. \tag{4.3.37}$$

Using (1.31) in (4.3.37), we get

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(8n+7)q^n \equiv 2f_5^2 f_{10} \pmod{2^2}, \tag{4.3.38}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(8n+7)q^n \equiv 2f_{20} \pmod{2^2}. \tag{4.3.39}$$

Congruence (4.3.21) follows by extracting the terms involving q^{4n+i} from both sides of (4.3.39).

Extracting the terms involving q^{4n} from (4.3.39) and then replacing q^4 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(32n+7)q^n \equiv 2f_5 \pmod{2^2}. \tag{4.3.40}$$

Congruence (4.3.22) follows by extracting the terms involving q^{5n+i} from both sides of (4.3.40).

Extracting the terms involving q^{5n} from (4.3.40) and then replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(160n+7)q^n \equiv 2f_1 \pmod{2^2}. \tag{4.3.41}$$

In view of congruences (4.3.36) and (4.3.41), we obtain (4.3.23). □

Theorem 4.3.5. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{A}_{4,1}^{5}\left(2\cdot 5^{2\alpha+3}n + \frac{14\cdot 5^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^2}, \tag{4.3.42}$$

$$\overline{A}_{4,1}^{5}(10n+5) \equiv 0 \pmod{2^2}, \tag{4.3.43}$$

$$\overline{A}_{4,1}^{5}(10n+9) \equiv 0 \pmod{2^2}, \tag{4.3.44}$$

$$\overline{A}_{4,1}^{5}(10(5n+i)+7) \equiv 0 \pmod{2^2}, \tag{4.3.45}$$

where i = 1, 2, 3, 4.

Proof. Employing (1.34) into (4.3.27), we get

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(2n+1)q^n \equiv 2f_5f_{25}^3\left(a^3-3a^2q+5q^3-3\frac{q^5}{a^2}-\frac{q^6}{a^3}\right) \pmod{2^2}. \tag{4.3.46}$$

Congruences (4.3.43) and (4.3.44) follow by extracting the terms involving q^{5n+2} and q^{5n+4} from both sides of (4.3.46).

Extracting the terms involving q^{5n+3} from (4.3.46), dividing by q^3 and then replacing q^5 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(10n+7)q^n &\equiv 10f_1f_5^3 \pmod{2^2}\\ &\equiv 2f_5^3f_{25}\left(a-q-\frac{q^2}{a}\right) \pmod{2^2}.\end{aligned} \tag{4.3.47}$$

Congruence (4.3.45) follows by extracting the terms involving q^{5n+i} from both sides of (4.3.47).

Extracting the terms involving q^{5n+1} from (4.3.47), dividing by q and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(50n+17)q^n \equiv 2f_1^3f_5 \pmod{2^2}. \tag{4.3.48}$$

In view of congruences (4.3.27) and (4.3.48), we obtain

$$\overline{A}_{4,1}^{5}(50n+17)q^n \equiv \overline{A}_{4,1}^{5}(2n+1) \pmod{2^2}. \tag{4.3.49}$$

Utilizing (4.3.49) and by mathematical induction on α, we get

$$\overline{A}_{4,1}^{5}\left(2\cdot 25^{\alpha+1}n+\frac{2\cdot 25^{\alpha+1}+1}{3}\right) \equiv \overline{A}_{4,1}^{5}(2n+1) \pmod{2^2}. \tag{4.3.50}$$

Utilizing (4.3.43) and (4.3.50), we get (4.3.42). □

Theorem 4.3.6. *For all $n \geq 0$,*

$$\overline{A}^5_{4,1}(32n+31) \equiv 0 \pmod{2^3}, \tag{4.3.51}$$

$$\sum_{n=0}^{\infty} \overline{A}^5_{4,1}(32n+15)q^n \equiv 4f_1f_{10} \pmod{2^3}. \tag{4.3.52}$$

Proof. Using (1.31) in (4.3.26), we obtain

$$\sum_{n=0}^{\infty} \overline{A}^5_{4,1}(2n+1)q^n \equiv 2\frac{f_1f_2f_{10}}{f_5} \pmod{2^3}. \tag{4.3.53}$$

Substituting (1.53) into (4.3.53), we arrive at

$$\sum_{n=0}^{\infty} \overline{A}^5_{4,1}(2n+1)q^n \equiv 2\frac{f_2^2f_8f_{20}^3}{f_4f_{10}^2f_{40}} - 2q\frac{f_2f_4^2f_{40}}{f_8f_{10}} \pmod{2^3}. \tag{4.3.54}$$

Extracting the terms involving q^{2n+1} from (4.3.54), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}^5_{4,1}(4n+3)q^n \equiv 6\frac{f_1f_2^2f_{20}}{f_4f_5} \pmod{2^3}. \tag{4.3.55}$$

Employing (1.53) in (4.3.55), we get

$$\sum_{n=0}^{\infty} \overline{A}^5_{4,1}(4n+3)q^n \equiv 6\frac{f_2^3f_8f_{20}^4}{f_4^2f_{10}^3f_{40}} - 6q\frac{f_2^2f_4f_{20}f_{40}}{f_8f_{10}^2} \pmod{2^3}. \tag{4.3.56}$$

Extracting the terms involving q^{2n+1} from (4.3.56), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}^5_{4,1}(8n+7)q^n \equiv 2\frac{f_1^2f_2f_{10}f_{20}}{f_4f_5^2} \pmod{2^3}. \tag{4.3.57}$$

Again substituting (1.53) in (4.3.57), we find that

$$\sum_{n=0}^{\infty} \overline{A}^5_{4,1}(16n+15)q^n \equiv 4\frac{f_1^2f_{10}^4}{f_5^4} \pmod{2^3}. \tag{4.3.58}$$

Using (1.31) in (4.3.58), we arrive at

$$\sum_{n=0}^{\infty}\overline{A}_{4,1}^{5}(16n+15)q^n \equiv 4f_2f_{20} \pmod{2^3}, \tag{4.3.59}$$

Congruence (4.3.51) follows by extracting the terms involving q^{2n+1} from (4.3.59).

Collecting the terms involving q^{2n} from (4.3.59) and replacing q^2 by q, we get (4.3.52). □

Theorem 4.3.7. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{A}_{4,1}^{5}\left(32\cdot 5^{2\alpha+2}(5n+i)+\frac{44\cdot 5^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^3}, \tag{4.3.60}$$

where i=3, 4.

$$\overline{A}_{4,1}^{5}(32(5n+i)+15) \equiv 0 \pmod{2^3}, \tag{4.3.61}$$

where i=3, 4.

$$\overline{A}_{4,1}^{5}(160(5n+j)+47) \equiv 0 \pmod{2^3}, \tag{4.3.62}$$

where j=1, 3.

Proof. Substituting (1.34) into (4.3.52), we find that

$$\sum_{n=0}^{\infty}\overline{A}_{4,1}^{5}(32n+15)q^n \equiv 4f_{10}f_{25}\left(a-q-\frac{q^2}{a}\right) \pmod{2^3}. \tag{4.3.63}$$

Congruence (4.3.61) follows by extracting the terms involving q^{5n+i} from both sides of (4.3.63).

Extracting the terms involving q^{5n+1} from (4.3.63), dividing by q and then replacing q^5 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty}\overline{A}_{4,1}^{5}(160n+47)q^n &\equiv 4f_2f_5 \pmod{2^3}\\ &\equiv 2f_5f_{50}\left(a(q^2)-q^2-\frac{q^4}{a(q^2)}\right) \pmod{2^3}.\end{aligned} \tag{4.3.64}$$

Congruence (4.3.62) follows by extracting the terms involving q^{5n+j} from both sides of (4.3.64).

Extracting the terms involving q^{5n+2} from (4.3.64), dividing by q^2 and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(800n+367)q^n \equiv 4f_1f_{10} \pmod{2^3}. \tag{4.3.65}$$

In view of congruences (4.3.52) and (4.3.65), we obtain

$$\overline{A}_{4,1}^{5}(800n+367)q^n \equiv \overline{A}_{4,1}^{5}(32n+15) \pmod{2^3}. \tag{4.3.66}$$

Utilizing (4.3.66) and by mathematical induction on α, we get

$$\overline{A}_{4,1}^{5}\left(32\cdot 25^{\alpha+1}n+\frac{44\cdot 25^{\alpha+1}+1}{3}\right) \equiv \overline{A}_{4,1}^{5}(32n+15) \pmod{2^3}. \tag{4.3.67}$$

Utilizing (4.3.61) and (4.3.67), we get (4.3.60). □

Theorem 4.3.8. *For any prime $p \geq 5$ with $\left(\frac{-10}{p}\right) = -1$, $\alpha \geq 1$ and $n \geq 0$,*

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}\left(32\cdot p^{2\alpha}n+\frac{44\cdot p^{2\alpha}+1}{3}\right)q^n \equiv 4f_1f_{10} \pmod{2^3}. \tag{4.3.68}$$

Proof. Define

$$\sum_{n=0}^{\infty} g(n)q^n = f_1f_{10} \pmod{2^3}. \tag{4.3.69}$$

Combining (4.3.52) and (4.3.69), we find that

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(32n+15)q^n \equiv 4\sum_{n=0}^{\infty} g(n)q^n \pmod{2^3}. \tag{4.3.70}$$

Now, we consider the congruence equation

$$\frac{3k^2+k}{2}+10\cdot\frac{3m^2+m}{2} \equiv \frac{11p^2-11}{24} \pmod{p}, \tag{4.3.71}$$

which is equivalent to

$$(6k+1)^2 + 10\cdot(6m+1)^2 \equiv 0 \pmod{p}.$$

where $\frac{-(p-1)}{2} \le k, m \le \frac{p-1}{2}$ and p is a prime such that $(\frac{-10}{p}) = -1$. Since $(\frac{-10}{p}) = -1$ for $p \ge 5$, the congruence relation (4.3.71) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute (1.36) into (4.3.69) and then extracting the terms in which the powers of q are congruent to $11\cdot\frac{p^2-1}{24}$ modulo p and then divide by $q^{11\cdot\frac{p^2-1}{24}}$, we find that

$$\sum_{n=0}^{\infty} g\left(pn + 11\cdot\frac{p^2-1}{24}\right)q^{pn} = f_p f_{10p}, \tag{4.3.72}$$

which implies that

$$\sum_{n=0}^{\infty} g\left(p^2n + 11\cdot\frac{p^2-1}{24}\right)q^{n} = f_1 f_{10} \tag{4.3.73}$$

and for $n \ge 0$,

$$g\left(p^2n + pi + 11\cdot\frac{p^2-1}{24}\right) = 0, \tag{4.3.74}$$

where i is an integer and $1 \le i \le p-1$. By induction, we see that for $n \ge 0$ and $\alpha \ge 0$,

$$g\left(p^{2\alpha}n + 11\cdot\frac{p^{2\alpha}-1}{24}\right) = g(n). \tag{4.3.75}$$

Replacing n by $p^{2\alpha}n + 11\cdot\frac{p^{2\alpha}-1}{24}$ in (4.3.70), we arrive at (4.3.52). □

Theorem 4.3.9. *For any prime $p \ge 5$ with $(\frac{-10}{p}) = -1$, $\alpha \ge 1$ and $n \ge 0$,*

$$\overline{A}^{5}_{4,1}\left(32\cdot p^{2\alpha+2}n + 32\cdot p^{2\alpha+1}i + \frac{44\cdot p^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^3}.$$

where $i = 1, 2, \ldots, p-1$.

Proof. Replacing n by $p^2n + pi + \frac{11\cdot p^2 - 11}{24}$ in (4.3.75) and using (4.3.74), we find that for $n \ge 0$ and $\alpha \ge 0$,

$$g\left(p^{2\alpha+2}n + p^{2\alpha+1}i + \frac{11\cdot p^{2\alpha+2}-11}{24}\right) = 0. \tag{4.3.76}$$

Comparing the coefficients of q^n from the both sides of (4.3.70), we see that for $n \ge 0$,

$$\overline{A}^{5}_{4,1}(32n+15) \equiv 4g(n) \pmod{2^3}. \tag{4.3.77}$$

The result follows from (4.3.76) and (4.3.77). □

Theorem 4.3.10. *For all* $n \geq 0$,

$$\overline{A}_{4,1}^{5}(16n+13) \equiv 0 \pmod{2^3}, \tag{4.3.78}$$

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(16n+5)q^n \equiv 4 f_2 f_5 \pmod{2^3}. \tag{4.3.79}$$

Proof. Equation (4.3.26) can be rewritten as

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(2n+1)q^n = 2 f_2^3 f_{10} \frac{f_1}{f_5 f_1^4}. \tag{4.3.80}$$

Substituting (1.41) and (1.53) into (4.3.80), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(2n+1)q^n = 2\frac{f_4^{13} f_{20}^3}{f_2^{10} f_8^3 f_{10}^2 f_{40}} - 8q^2 \frac{f_4^4 f_8^3 f_{40}}{f_2^7 f_{10}} - 2q \frac{f_4^{16} f_{40}}{f_2^{11} f_8^5 f_{10}} + 8q \frac{f_4 f_8^5 f_{20}^3}{f_2^6 f_{10}^2 f_{40}}. \tag{4.3.81}$$

Collecting the terms involving q^{2n} from (4.3.81) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(4n+1)q^n = 2\frac{f_2^{13} f_{10}^3}{f_1^{10} f_4^3 f_5^2 f_{20}} - 8q \frac{f_2^4 f_4^3 f_{20}}{f_1^7 f_5}. \tag{4.3.82}$$

Using (1.31) in (4.3.82), we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(4n+1)q^n \equiv 2\frac{f_2 f_4 f_5^2 f_{10}}{f_1^2 f_{20}} \pmod{8}. \tag{4.3.83}$$

Employing (1.52) into (4.3.83), we arrive at

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(4n+1)q^n \equiv 2\frac{f_4 f_8^2 f_{10} f_{20}^3}{f_2^3 f_{40}^2} + 2q^2 \frac{f_4^7 f_{10}^3 f_{40}^2}{f_2^5 f_8^2 f_{20}^3} + 4q \frac{f_4^4 f_{10}^2}{f_2^4} \pmod{8}. \tag{4.3.84}$$

Extracting the terms involving q^{2n+1} from (4.3.84), dividing by q and then replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(8n+5)q^n \equiv 4\frac{f_2^4 f_5^2}{f_1^4} \pmod{8}. \tag{4.3.85}$$

Using (1.31) in (4.3.85), we get

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(8n+5)q^n \equiv 4f_4 f_{10} \pmod{8}. \tag{4.3.86}$$

Congruence (4.3.78) follows by extracting the terms involving q^{2n+1} from (4.3.86).

Collecting the terms involving q^{2n} from (4.3.86) and replacing q^2 by q, we get (4.3.79). □

Theorem 4.3.11. *For all $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{A}_{4,1}^{5}\left(16 \cdot 5^{2\alpha+2}(5n+i) + \frac{14 \cdot 5^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^3}, \tag{4.3.87}$$

where i=1, 3.

$$\overline{A}_{4,1}^{5}(16(5n+i)+5) \equiv 0 \pmod{2^3}, \tag{4.3.88}$$

where i=1, 3.

$$\overline{A}_{4,1}^{5}(80(5n+j)+37) \equiv 0 \pmod{2^3}, \tag{4.3.89}$$

where j=3, 4.

Proof. Employing (1.34) into (4.3.79), we arrive at

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(16n+5)q^n \equiv 4f_5 f_{50}\left(a(q^2) - q^2 - \frac{q^4}{a(q^2)}\right) \pmod{2^3}. \tag{4.3.90}$$

Congruence (4.3.88) follows by extracting the terms involving q^{5n+i} from both sides of (4.3.90).

Extracting the terms involving q^{5n+2} from (4.3.90), dividing by q^2 and then replacing q^5 by q, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(80n+37)q^n &\equiv 4f_1 f_{10} \pmod{2^3} \\ &\equiv 4f_{10} f_{25}\left(a - q - \frac{q^2}{a}\right) \pmod{2^3}. \end{aligned} \tag{4.3.91}$$

Congruence (4.3.89) follows by extracting the terms involving q^{5n+j} from both sides of (4.3.91).

Extracting the terms involving q^{5n+1} from (4.3.91), dividing by q and then replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(400n+117)q^n \equiv 4f_2f_5 \pmod{2^3}. \tag{4.3.92}$$

In view of congruences (4.3.79) and (4.3.92), we obtain

$$\overline{A}_{4,1}^{5}(400n+117)q^n \equiv \overline{A}_{4,1}^{5}(16n+5) \pmod{2^3}. \tag{4.3.93}$$

Utilizing (4.3.93) and by mathematical induction on α, we get

$$\overline{A}_{4,1}^{5}\left(16\cdot 25^{\alpha+1}n+\frac{14\cdot 25^{\alpha+1}+1}{3}\right) \equiv \overline{A}_{4,1}^{5}(16n+5) \pmod{2^3}. \tag{4.3.94}$$

Using (4.3.88) in (4.3.94), we get (4.3.87). □

Theorem 4.3.12. *For any prime $p \geq 5$ with $(\frac{-10}{p}) = -1$, $\alpha \geq 1$ and $n \geq 0$,*

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}\left(16\cdot p^{2\alpha}n+\frac{14\cdot p^{2\alpha}+1}{3}\right)q^n \equiv 4f_2f_5 \pmod{2^3}. \tag{4.3.95}$$

Proof. Define

$$\sum_{n=0}^{\infty} h(n)q^n = f_2f_5 \pmod{2^3}. \tag{4.3.96}$$

Combining (4.3.79) and (4.3.96), we find that

$$\sum_{n=0}^{\infty} \overline{A}_{4,1}^{5}(16n+5)q^n \equiv 4\sum_{n=0}^{\infty} h(n)q^n \pmod{2^3}. \tag{4.3.97}$$

Now, we consider the congruence equation

$$2\cdot\frac{3k^2+k}{2}+5\cdot\frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod{p}, \tag{4.3.98}$$

which is equivalent to

$$(12k+2)^2 + 10\cdot(6m+1)^2 \equiv 0 \pmod{p}.$$

where $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$ and p is a prime such that $(\frac{-10}{p}) = -1$. Since $(\frac{-10}{p}) = -1$ for $p \geq 5$, the congruence relation (4.3.98) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute (1.36) into (4.3.96) and then extracting the terms in which the powers of q are congruent to $7\cdot\frac{p^2-1}{24}$ modulo p and then divide by $q^{7\cdot\frac{p^2-1}{24}}$, we find that

$$\sum_{n=0}^{\infty} h\left(pn + 7\cdot\frac{p^2-1}{24}\right) q^{pn} = f_{2p} f_{5p},$$

which implies that

$$\sum_{n=0}^{\infty} h\left(p^2 n + 7\cdot\frac{p^2-1}{24}\right) q^{n} = f_2 f_5 \tag{4.3.99}$$

and for $n \geq 0$,

$$h\left(p^2 n + pi + 7\cdot\frac{p^2-1}{24}\right) = 0, \tag{4.3.100}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$h\left(p^{2\alpha} n + 7\cdot\frac{p^{2\alpha}-1}{24}\right) = h(n). \tag{4.3.101}$$

Replacing n by $p^{2\alpha} n + 7\cdot\frac{p^{2\alpha}-1}{24}$ in (4.3.97), we arrive at (4.3.95). □

Theorem 4.3.13. *For any prime $p \geq 5$ with $(\frac{-10}{p}) = -1$, $\alpha \geq 1$ and $n \geq 0$,*

$$\overline{A}_{4,1}^{5}\left(16\cdot p^{2\alpha+2} n + 16\cdot p^{2\alpha+1} i + \frac{14\cdot p^{2\alpha+2}+1}{3}\right) \equiv 0 \pmod{2^3}.$$

where $i = 1, 2, \ldots, p-1$.

Proof. Replacing n by $p^2 n + pi + \frac{7\cdot p^2-7}{24}$ in (4.3.101) and using (4.3.100), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$h\left(p^{2\alpha+2} n + p^{2\alpha+1} i + \frac{7\cdot p^{2\alpha+2}-7}{24}\right) = 0. \tag{4.3.102}$$

Comparing the coefficients of q^n from the both sides of (4.3.97), we see that for $n \geq 0$,

$$\overline{A}_{4,1}^{5}(16n+5) \equiv 4h(n) \pmod{2^3}. \tag{4.3.103}$$

The result follows from (4.3.102) and (4.3.103). □

4.4 Some new congruences for Andrews' singular overpartition pairs

In this section, Mahadeva Naika and Shivashankar [61] have defined the Andrews' singular overpatition pairs of n. Let $\overline{C}_{i,j}^{\delta}(n)$ denote the number of Andrews' singular overpatition pairs of n in which no part is divisible by δ and only parts congruent to $\pm i$, $\pm j$ modulo δ may be overlined. Andrews' singular overpatition pair π of n is a pair of Andrews' singular overpatitions (ν_1, ν_2) such that the sum of all of the parts is n. For $\delta \geq 3$ and $1 \leq i, j \leq \lfloor \frac{\delta}{2} \rfloor$, the generating function for $\overline{C}_{i,j}^{\delta}(n)$ is

$$\sum_{n=0}^{\infty} \overline{C}_{i,j}^{\delta}(n) q^n = \frac{f(q^i, q^{\delta-i}) f(q^j, q^{\delta-j})}{(q;q)_\infty^2}. \tag{4.4.1}$$

4.4.1 Infinite family of congruence modulo 27 for $\overline{C}_{1,2}^{6}(n)$

Theorem 4.4.1. *For any $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{C}_{1,2}^{6}\left(4^{\alpha+2} n + \frac{4^{\alpha+2}-1}{3}\right) \equiv 7^{\alpha+1} \cdot \overline{C}_{1,2}^{6}(4n+1) \pmod{27}. \tag{4.4.2}$$

Proof. Setting $i = 1$, $j = 2$ and $\delta = 6$ in (4.4.1), we see that

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(n) q^n = \frac{f(q, q^5) f(q^2, q^4)}{(q;q)_\infty^2}. \tag{4.4.3}$$

By the definition of $f(a, b)$ and the well-known Jacobi triple product identity, we get

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(n) q^n = \frac{f_2 f_3 f_6}{f_1^3}. \tag{4.4.4}$$

Substituting (1.43) into (4.4.4), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(n) q^n = f_2 f_6 \left(\frac{f_4^6 f_6^3}{f_2^9 f_{12}^2} + 3q \frac{f_4^2 f_6 f_{12}^2}{f_2^7} \right). \tag{4.4.5}$$

Equating odd parts of the above equation, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(2n+1)q^n = 3\frac{f_2^2 f_3^2 f_6^2}{f_1^6}. \tag{4.4.6}$$

Employing (1.43) into (4.4.6), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(2n+1)q^n = 3\frac{f_4^{12} f_6^8}{f_2^{16} f_{12}^4} + 18q\frac{f_4^8 f_6^6}{f_2^{14}} + 27q^2\frac{f_4^4 f_6^4 f_{12}^4}{f_2^{12}}. \tag{4.4.7}$$

Extracting the terms involving q^{2n} from both sides of (4.4.7), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(4n+1)q^n = 3\frac{f_2^{12} f_3^8}{f_1^{16} f_6^4} + 27q\frac{f_2^4 f_3^4 f_6^4}{f_1^{12}}, \tag{4.4.8}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(4n+1)q^n \equiv 3\frac{f_2^{12} f_3^8}{f_1^{16} f_6^4} \pmod{27}. \tag{4.4.9}$$

Invoking (1.31) in (4.4.9), we see that

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(4n+1)q^n \equiv 3\frac{f_1^2 f_2^3 f_3^2}{f_6} \pmod{27}. \tag{4.4.10}$$

Substituting (1.49) into (4.4.10), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(4n+1)q^n \equiv 3\frac{f_2^5 f_8^4 f_{12}^8}{f_4^4 f_6^3 f_{24}^4} + 3q^2\frac{f_2 f_4^8 f_6 f_{24}^4}{f_8^4 f_{12}^4} - 6q\frac{f_2^3 f_4^2 f_{12}^2}{f_6} \pmod{27}. \tag{4.4.11}$$

Extracting the terms involving q^{2n+1} from (4.4.11), dividing by q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(8n+5)q^n \equiv 21\frac{f_1^3 f_2^2 f_6^2}{f_3} \pmod{27}. \tag{4.4.12}$$

Employing (1.45) into (4.4.12), the equation reduces to

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(8n+5)q^n \equiv 21\frac{f_2^2 f_4^3 f_6^2}{f_{12}} + 18q\frac{f_2^4 f_{12}^3}{f_4} \pmod{27}, \tag{4.4.13}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{C}_{1,2}^{6}(16n+5)q^n \equiv 21\frac{f_1^2 f_2^3 f_3^2}{f_6} \pmod{27}. \tag{4.4.14}$$

In view of congruences (4.4.10) and (4.4.14), we see that

$$\overline{C}_{1,2}^{6}(16n+5) \equiv 7 \cdot \overline{C}_{1,2}^{6}(4n+1) \pmod{27}. \tag{4.4.15}$$

Using the above relation and by induction on α, we arrive at (4.4.2). □

4.4.2 Congruences modulo 4 for $\overline{C}_{1,5}^{12}(n)$

Theorem 4.4.2. *For all* $\alpha \geq 0$ *and* $n \geq 0$,

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(3n)q^n \equiv \frac{f_1 f_3 f_6}{f_2} \pmod{4}, \tag{4.4.16}$$

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(3n+1)q^n \equiv 3\frac{f_1^2 f_6^4}{f_2^2 f_3^2} \pmod{4}, \tag{4.4.17}$$

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(3n+2)q^n \equiv 3\frac{f_1 f_3 f_{12}^3}{f_4 f_6^2} \pmod{4}, \tag{4.4.18}$$

$$\overline{C}_{1,5}^{12}(3^{\alpha+1}n + 3^{\alpha+1} - 1) \equiv 3^{\alpha+1} \cdot \overline{C}_{1,5}^{12}(n) \pmod{4}. \tag{4.4.19}$$

Proof. Setting $i = 1$, $j = 5$ and $\delta = 12$ in (4.4.1), we find that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(n)q^n = \frac{f(q, q^{11}) f(q^5, q^7)}{(q;q)_\infty^2}$$

$$= \frac{f_2^2 f_3 f_{12}^3}{f_1^3 f_4 f_6^2}. \tag{4.4.20}$$

Invoking (1.31) in (4.4.20), we see that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(n)q^n \equiv \frac{f_1 f_3 f_{12}^3}{f_4 f_6^2} \pmod{4}. \tag{4.4.21}$$

Substituting (1.77) into (4.4.21), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(n)q^n \equiv \frac{f_3 f_9 f_{18}}{f_6} - q\frac{f_3^2 f_{18}^4}{f_6^2 f_9^2} - q^2\frac{f_3 f_9 f_{36}^3}{f_{12} f_{18}^2} \pmod 4. \tag{4.4.22}$$

Extracting the terms involving q^{3n}, q^{3n+1} and q^{3n+2} from the above equation, we obtain respectively (4.4.16), (4.4.17) and (4.4.18).

In view of congruences (4.4.18) and (4.4.21), we deduce that

$$\overline{C}_{1,5}^{12}(3n+2) \equiv 3 \cdot \overline{C}_{1,5}^{12}(n) \pmod 4. \tag{4.4.23}$$

Using the above relation and by induction on α, we arrive at (4.4.19). □

Theorem 4.4.3. *For all integers $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{C}_{1,5}^{12}(12n+6) \equiv 0 \pmod 4, \tag{4.4.24}$$

$$\overline{C}_{1,5}^{12}(48n+27) \equiv 0 \pmod 4, \tag{4.4.25}$$

$$\overline{C}_{1,5}^{12}(96n+87) \equiv 0 \pmod 4, \tag{4.4.26}$$

$$\overline{C}_{1,5}^{12}(3 \cdot 4^{\alpha+3}n + 7 \cdot 4^{\alpha+2} - 1) \equiv 0 \pmod 4. \tag{4.4.27}$$

Proof. Substituting (1.49) in (4.4.16), we find that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(3n)q^n \equiv \frac{f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - q\frac{f_4^4 f_6^2 f_{24}^2}{f_2^2 f_8^2 f_{12}^2} \pmod 4. \tag{4.4.28}$$

Collecting the terms involving q^{2n} from (4.4.28) and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(6n)q^n \equiv \frac{f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod 4. \tag{4.4.29}$$

Using (1.31) in (4.4.29), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(6n)q^n \equiv f_2^2 \pmod 4. \tag{4.4.30}$$

Extracting the terms involving q^{2n+1} from (4.4.30), we obtain (4.4.24).

Extracting the terms involving q^{2n+1} from (4.4.28), dividing by q and replacing q^2

by q, we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(6n+3)q^n \equiv 3\frac{f_2^4 f_3^2 f_{12}^2}{f_1^2 f_4^2 f_6^2} \pmod 4. \tag{4.4.31}$$

Invoking (1.31) in (4.4.31), we get

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(6n+3)q^n \equiv 3\frac{f_3^2 f_6^2}{f_1^2} \pmod 4. \tag{4.4.32}$$

Employing (1.47) into (4.4.32), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(6n+3)q^n \equiv 3\frac{f_4^4 f_6^3 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^4 f_8 f_{24}}{f_2^4 f_{12}} \pmod 4. \tag{4.4.33}$$

Collecting the even terms of the above equation, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+3)q^n \equiv 3\frac{f_2^4 f_3^3 f_6^2}{f_1^5 f_4 f_{12}} \pmod 4. \tag{4.4.34}$$

Using (1.31) in (4.4.34), we see that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+3)q^n \equiv 3\frac{f_2^2 f_3^3 f_6^2}{f_1 f_4 f_{12}} \pmod 4. \tag{4.4.35}$$

Substituting (1.42) into (4.4.35), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+3)q^n \equiv 3\frac{f_4^2 f_6^4}{f_{12}^2} + 3q\frac{f_2^2 f_6^2 f_{12}^2}{f_4^2} \pmod 4, \tag{4.4.36}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(24n+3)q^n \equiv 3\frac{f_2^2 f_3^4}{f_6^2} \pmod 4. \tag{4.4.37}$$

Invoking (1.31) in (4.4.37), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(24n+3)q^n \equiv 3f_2^2 \pmod 4. \tag{4.4.38}$$

Extracting the terms involving q^{2n+1} from the above equation, we get (4.4.25).

Extracting the terms involving q^{2n+1} from (4.4.36), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}^{12}_{1,5}(24n+15)q^n \equiv 3\frac{f_1^2 f_3^2 f_6^2}{f_2^2} \pmod 4. \tag{4.4.39}$$

Using (1.31) in (4.4.39), we see that

$$\sum_{n=0}^{\infty} \overline{C}^{12}_{1,5}(24n+15)q^n \equiv 3\frac{f_3^2 f_6^2}{f_1^2} \pmod 4. \tag{4.4.40}$$

Employing (1.47) into (4.4.40), we have

$$\sum_{n=0}^{\infty} \overline{C}^{12}_{1,5}(24n+15)q^n \equiv 3\frac{f_4^4 f_6^3 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^4 f_8 f_{24}}{f_2^4 f_{12}} \pmod 4, \tag{4.4.41}$$

which implies,

$$\sum_{n=0}^{\infty} \overline{C}^{12}_{1,5}(48n+15)q^n \equiv 3\frac{f_2^4 f_3^3 f_6^2}{f_1^5 f_4 f_{12}} \pmod 4. \tag{4.4.42}$$

Invoking (1.31) in (4.4.42), we get

$$\sum_{n=0}^{\infty} \overline{C}^{12}_{1,5}(48n+15)q^n \equiv 3\frac{f_2^2 f_3^3 f_6^2}{f_1 f_4 f_{12}} \pmod 4. \tag{4.4.43}$$

In view of congruences (4.4.35) and (4.4.43), we obtain

$$\overline{C}^{12}_{1,5}(48n+15) \equiv \overline{C}^{12}_{1,5}(12n+3) \pmod 4. \tag{4.4.44}$$

Using the above relation and by induction on α, we arrive at

$$\overline{C}^{12}_{1,5}(3\cdot 4^{\alpha+2}n + 4\cdot 4^{\alpha+2} - 1) \equiv \overline{C}^{12}_{1,5}(12n+3) \pmod 4. \tag{4.4.45}$$

Using the congruence (4.4.25) in (4.4.45), we obtain (4.4.27).

Extracting the terms involving q^{2n+1} from (4.4.41), dividing by q and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}^{12}_{1,5}(48n+39)q^n \equiv 2\frac{f_2 f_3^4 f_4 f_{12}}{f_1^4 f_6} \pmod 4. \tag{4.4.46}$$

Invoking (1.31) in (4.4.46), we get

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(48n+39)q^n \equiv 2f_2f_6f_{12} \pmod 4. \tag{4.4.47}$$

Congruence (4.4.26) follows by extracting the terms involving q^{2n+1} from the above equation. □

Theorem 4.4.4. *For all $\alpha \geq 0$ and $n \geq 0$,*

$$\overline{C}_{1,5}^{12}(24n+13) \equiv 0 \pmod 4, \tag{4.4.48}$$
$$\overline{C}_{1,5}^{12}(24n+3) \equiv \overline{C}_{1,5}^{12}(12n+1) \pmod 4, \tag{4.4.49}$$
$$\overline{C}_{1,5}^{12}(24n+15) \equiv \overline{C}_{1,5}^{12}(12n+7) \pmod 4. \tag{4.4.50}$$

Proof. Employing (1.48) into (4.4.17), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(3n+1)q^n \equiv 3\frac{f_4^2f_{12}^4}{f_2f_6f_8f_{24}} + 2q\frac{f_6^3f_8f_{12}f_{24}}{f_4} \pmod 4. \tag{4.4.51}$$

Extracting the terms involving q^{2n} from (4.4.51) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(6n+1)q^n \equiv 3\frac{f_2^2f_6^4}{f_1f_3f_4f_{12}} \pmod 4. \tag{4.4.52}$$

Substituting (1.50) into (4.4.52), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(6n+1)q^n \equiv 3\frac{f_8^2f_{12}^4}{f_4^2f_{24}^2} + 3q\frac{f_4^4f_6^2f_{24}^2}{f_2^2f_8^2f_{12}^2} \pmod 4. \tag{4.4.53}$$

Extracting the terms involving q^{2n} from (4.4.53) and replacing q^2 by q, we deduce that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+1)q^n \equiv 3\frac{f_4^2f_6^4}{f_2^2f_{12}^2} \pmod 4. \tag{4.4.54}$$

Invoking (1.31) in (4.4.54), we get

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+1)q^n \equiv 3f_2^2 \pmod 4. \tag{4.4.55}$$

Extracting the terms involving q^{2n+1} from (4.4.55), we obtain (4.4.48) and combining (4.4.38) and (4.4.55), we get (4.4.49).

From the equation (4.4.53), which implies that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+7)q^n \equiv 3\frac{f_2^4 f_3^2 f_{12}^2}{f_1^2 f_4^2 f_6^2} \pmod 4. \tag{4.4.56}$$

Using (1.31) in (4.4.56), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n+7)q^n \equiv 3\frac{f_3^2 f_6^2}{f_1^2} \pmod 4. \tag{4.4.57}$$

Combining (4.4.40) and (4.4.57), we arrive at (4.4.50). □

Theorem 4.4.5. *For each $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{C}_{1,5}^{12}(12\cdot 25^{\alpha+1}n + 25^{\alpha+1} - 1) \equiv \overline{C}_{1,5}^{12}(12n) \pmod 4, \tag{4.4.58}$$

$$\overline{C}_{1,5}^{12}(60(5n+i)+24) \equiv 0 \pmod 4, \tag{4.4.59}$$

where $i = 1, 2, 3, 4$.

Proof. From the equation (4.4.30), we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n)q^n \equiv f_1^2 \pmod 4. \tag{4.4.60}$$

Employing (1.34) in the above equation and then extracting the terms containing q^{5n+2}, dividing by q^2 and replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(60n+24)q^n \equiv f_5^2 \pmod 4, \tag{4.4.61}$$

which yields

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(300n+24)q^n \equiv f_1^2 \equiv \sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(12n)q^n \pmod 4. \tag{4.4.62}$$

By induction on α, we obtain (4.4.58).

The congruence (4.4.59) follows by extracting the terms involving q^{5n+i} for $i = 1, 2, 3, 4$ from both sides of (4.4.61). □

Theorem 4.4.6. *Let p be a prime ≥ 5, $\left(\frac{-4}{p}\right) = -1$. Then for all integers $\alpha \geq 1$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(192p^{2\alpha}n + 39p^{2\alpha} - 1)q^n \equiv 2f_1 f_4 \pmod{4}. \tag{4.4.63}$$

Proof. Extracting the terms involving q^{2n} from (4.4.47) and replacing q^2 by q we have

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(96n + 39)q^n \equiv 2f_1 f_3 f_6 \pmod{4}. \tag{4.4.64}$$

Substituting (1.49) into (4.4.64), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(96n + 39)q^n \equiv 2\frac{f_2 f_8^2 f_{12}^4}{f_4^2 f_{24}^2} - 2q\frac{f_4^4 f_6^2 f_{24}^2}{f_2 f_8^2 f_{12}^2} \pmod{4}. \tag{4.4.65}$$

Extracting the even terms in the above equation, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(192n + 39)q^n \equiv 2\frac{f_1 f_4^2 f_6^4}{f_2^2 f_{12}^2} \pmod{4}. \tag{4.4.66}$$

Using (1.31) in (4.4.66), we see that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(192n + 39)q^n \equiv 2f_1 f_4 \pmod{4}. \tag{4.4.67}$$

Define

$$\sum_{n=0}^{\infty} f(n)q^n = f_1 f_4. \tag{4.4.68}$$

Combining (4.4.67) and (4.4.68), we find that

$$\sum_{n=0}^{\infty} \overline{C}_{1,5}^{12}(192n + 39)q^n \equiv 2\sum_{n=0}^{\infty} f(n)q^n \pmod{4}. \tag{4.4.69}$$

For a prime, $p \geq 5$ or $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$, consider

$$\frac{3k^2+k}{2} + 4 \cdot \frac{3m^2+m}{2} \equiv \frac{5p^2-5}{24} \pmod{p}, \tag{4.4.70}$$

therefore,

$$(6k+1)^2 + 4 \cdot (6m+1)^2 \equiv 0 \pmod{p},$$

Since $\left(\frac{-4}{p}\right) = -1$ the congruence relation (4.4.70) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute Lemma (1.36) into (4.4.68) and then extract the terms in which the powers of q are congruent to $\frac{5p^2-5}{24}$ modulo p and then divide by $q^{\frac{5p^2-5}{24}}$, we find that

$$\sum_{n=0}^{\infty} f\left(pn + \frac{5p^2-5}{24}\right) q^{pn} = f_{p^2} f_{4p^2},$$

which implies that

$$\sum_{n=0}^{\infty} f\left(p^2 n + \frac{5p^2-5}{24}\right) q^{n} = f_1 f_4 \tag{4.4.71}$$

and for $n \geq 0$,

$$f\left(p^2 n + pi + \frac{5p^2-5}{24}\right) = 0, \tag{4.4.72}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha} n + \frac{5p^{2\alpha}-5}{24}\right) = f(n). \tag{4.4.73}$$

Replacing n by $p^{2\alpha} n + \frac{5p^{2\alpha}-5}{24}$ in (4.4.69), we arrive at (4.4.63). □

Theorem 4.4.7. *Let p be a prime ≥ 5, $\left(\frac{-4}{p}\right) = -1$. Then for all integers $\alpha \geq 0$, and $n \geq 0$,*

$$\overline{C}^{12}_{1,5}(192p^{2\alpha+2}n + 192p^{2\alpha+1}i + 40p^{2\alpha+2} - 1) \equiv 0 \pmod{4}, \tag{4.4.74}$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. Replacing n by $p^2 n + pi + \frac{5p^2-5}{24}$ in (4.4.73) and using (4.4.72), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$f\left(p^{2\alpha+2} n + p^{2\alpha+1} i + \frac{5p^{\alpha+2}-5}{24}\right) = 0. \tag{4.4.75}$$

Comparing coefficients of q^n from both sides of (4.4.69), we see that for $n \geq 0$,

$$\overline{C}_{1,5}^{12}(192n+39) \equiv 2f(n) \pmod 4. \tag{4.4.76}$$

The required result follows from (4.4.75) and (4.4.76). □

4.4.3 Infinite families of congruences modulo 4 for $\overline{C}_{3,3}^{9}(n)$

Theorem 4.4.8. *Let p be a prime ≥ 5, $\left(\frac{-6}{p}\right) = -1$. Then for all integers $\alpha \geq 1$, and $n \geq 0$,*

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}\left(4p^{2\alpha}n + \frac{7p^{2\alpha}-1}{6}\right)q^n \equiv 2f_1f_6 \pmod 4. \tag{4.4.77}$$

Proof. Setting $i = 3$, $j = 3$ and $\delta = 9$ in (4.4.1), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}(n)q^n = \frac{f(q^3,q^6)f(q^3,q^6)}{(q;q)_\infty^2}.$$

After q-product manipulation, we see that

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}(n)q^n = \frac{f_6^2 f_9^4}{f_1^2 f_3^2 f_{18}^2}. \tag{4.4.78}$$

Invoking (1.31) in (4.4.78), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}(n)q^n \equiv \frac{f_3^2}{f_1^2} \pmod 4. \tag{4.4.79}$$

Employing (1.47) into (4.4.79), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}(n)q^n \equiv \frac{f_4^4 f_6 f_{12}^2}{f_2^5 f_8 f_{24}} + 2q\frac{f_4 f_6^2 f_8 f_{24}}{f_2^4 f_{12}} \pmod 4. \tag{4.4.80}$$

Extracting the odd terms of the above equation, we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}(2n+1)q^n \equiv 2\frac{f_2 f_3^2 f_4 f_{12}}{f_1^4 f_6} \pmod 4. \tag{4.4.81}$$

Using (1.31) in (4.4.81), we get

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}(2n+1)q^n \equiv 2f_2f_{12} \pmod 4, \tag{4.4.82}$$

which implies,

$$\overline{C}_{3,3}^{9}(4n+3) \equiv 0 \pmod 4. \tag{4.4.83}$$

Extracting the even terms from (4.4.82), we have

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}(4n+1)q^n \equiv 2f_1f_6 \pmod 4. \tag{4.4.84}$$

Define

$$\sum_{n=0}^{\infty} g(n)q^n = f_1f_6. \tag{4.4.85}$$

Combining (4.4.84) and (4.4.85), we find that

$$\sum_{n=0}^{\infty} \overline{C}_{3,3}^{9}(4n+1)q^n \equiv 2\sum_{n=0}^{\infty} g(n)q^n \pmod 4. \tag{4.4.86}$$

For a prime, $p \geq 5$ or $\frac{-(p-1)}{2} \leq k, m \leq \frac{p-1}{2}$, consider

$$\frac{3k^2+k}{2} + 6\cdot\frac{3m^2+m}{2} \equiv \frac{7p^2-7}{24} \pmod p, \tag{4.4.87}$$

therefore,

$$(6k+1)^2 + 6\cdot(6m+1)^2 \equiv 0 \pmod p,$$

Since $\left(\frac{-6}{p}\right) = -1$ the congruence relation (4.4.87) holds if and only if both $k = m = \frac{\pm p-1}{6}$. Therefore, if we substitute Lemma (1.36) into (4.4.85) and then extract the terms in which the powers of q are congruent to $\frac{7p^2-7}{24}$ modulo p and then divide by $q^{\frac{7p^2-7}{24}}$, we find that

$$\sum_{n=0}^{\infty} g\left(pn + \frac{7p^2-7}{24}\right)q^{pn} = f_{p^2}f_{6p^2},$$

which implies that

$$\sum_{n=0}^{\infty} g\left(p^2n + \frac{7p^2-7}{24}\right)q^n = f_1f_6 \tag{4.4.88}$$

and for $n \geq 0$,

$$g\left(p^2 n + pi + \frac{7p^2 - 7}{24}\right) = 0, \tag{4.4.89}$$

where i is an integer and $1 \leq i \leq p-1$. By induction, we see that for $n \geq 0$ and $\alpha \geq 0$,

$$g\left(p^{2\alpha} n + \frac{7p^{2\alpha} - 7}{24}\right) = g(n). \tag{4.4.90}$$

Replacing n by $p^{2\alpha} n + \frac{7p^{2\alpha}-7}{24}$ in (4.4.86), we arrive at (4.4.77). □

Theorem 4.4.9. *Let p be a prime ≥ 5, $\left(\frac{-6}{p}\right) = -1$. Then for all integers $\alpha \geq 0$, and $n \geq 0$,*

$$\overline{C}_{3,3}^{9}\left(4p^{2\alpha+2} n + 4p^{2\alpha+1} i + \frac{7p^{2\alpha+2} - 1}{6}\right) \equiv 0 \pmod{4}, \tag{4.4.91}$$

where i is an integer and $1 \leq i \leq p-1$.

Proof. Replacing n by $p^2 n + pi + \frac{7p^2-7}{24}$ in (4.4.90) and using (4.4.89), we find that for $n \geq 0$ and $\alpha \geq 0$,

$$g\left(p^{2\alpha+2} n + p^{2\alpha+1} i + \frac{7p^{\alpha+2} - 7}{24}\right) = 0. \tag{4.4.92}$$

Comparing coefficients of q^n from both sides of (4.4.86), we see that for $n \geq 0$,

$$\overline{C}_{3,3}^{9}(4n+1) \equiv 2g(n) \pmod{4}. \tag{4.4.93}$$

The required result follows from (4.4.92) and (4.4.93). □

4.4.4 Infinite families of congruences modulo 4 for $\overline{C}_{5,5}^{15}(n)$

Theorem 4.4.10. *For each $n \geq 0$ and $\alpha \geq 0$,*

$$\overline{C}_{5,5}^{15}(16n+9) \equiv 0 \pmod{4}, \tag{4.4.94}$$

$$\overline{C}_{5,5}^{15}(20n+3) \equiv \overline{C}_{5,5}^{15}(8n+1) \pmod{4}, \tag{4.4.95}$$

$$\overline{C}_{5,5}^{15}(10n+3) \equiv \overline{C}_{5,5}^{15}(4n+1) \pmod{4}, \tag{4.4.96}$$

$$\overline{C}_{5,5}^{15}\left(2 \cdot 4^{\alpha+1} n + \frac{4^{\alpha+2} - 1}{3}\right) \equiv \overline{C}_{5,5}^{15}(2n+1) \pmod{4}, \tag{4.4.97}$$

$$\overline{C}_{5,5}^{15}\left(4\cdot 5^{5\alpha+6}(5n+i)+\frac{2\cdot 5^{5\alpha+6}-1}{3}\right)\equiv 0 \pmod 4, \tag{4.4.98}$$

where i=1, 2, 3, 4.

$$\overline{C}_{5,5}^{15}\left(2\cdot 5^{2\alpha+4}(5n+j)+\frac{4\cdot 5^{2\alpha+4}-1}{3}\right)\equiv 0 \pmod 4, \tag{4.4.99}$$

where j=3, 4.

Proof. Putting $i=5$, $j=5$ and $\delta=15$ in (4.4.1), we find that

$$\sum_{n=0}^{\infty}\overline{C}_{5,5}^{15}(n)q^n=\frac{f(q^5,q^{10})^2}{(q;q)_\infty^2}.$$

After q-product manipulation, we see that

$$\sum_{n=0}^{\infty}\overline{C}_{5,5}^{15}(n)q^n=\frac{f_{10}^2f_{15}^4}{f_1^2f_5^2f_{30}^2}. \tag{4.4.100}$$

Invoking (1.31) in (4.4.100), we have

$$\sum_{n=0}^{\infty}\overline{C}_{5,5}^{15}(n)q^n\equiv\frac{f_5^2}{f_1^2} \pmod 4. \tag{4.4.101}$$

Substituting (1.52) into (4.4.101), we get

$$\sum_{n=0}^{\infty}\overline{C}_{5,5}^{15}(n)q^n\equiv\left(\frac{f_8f_{20}^2}{f_2^2f_{40}}+q\frac{f_4^3f_{10}f_{40}}{f_2^3f_8f_{20}}\right)^2 \pmod 4, \tag{4.4.102}$$

which implies that

$$\sum_{n=0}^{\infty}\overline{C}_{5,5}^{15}(2n+1)q^n\equiv 2f_1f_5^3 \pmod 4. \tag{4.4.103}$$

Employing (1.58) into (4.4.103), we obtain

$$\sum_{n=0}^{\infty}\overline{C}_{5,5}^{15}(2n+1)q^n\equiv 2f_2^3f_{10}+2q\frac{f_2^2f_{10}^2f_{20}}{f_4} \pmod 4. \tag{4.4.104}$$

Extracting the terms involving q^{2n} from (4.4.104) and replacing q^2 by q, we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(4n+1)q^n \equiv 2f_1^3 f_5 \pmod 4 \tag{4.4.105}$$

Substituting (1.59) into (4.4.105), the equation reduces to

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(4n+1)q^n \equiv 2\frac{f_2^2 f_4 f_{10}^2}{f_{20}} + 2q f_2 f_{10}^3 \pmod 4, \tag{4.4.106}$$

which implies that

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(8n+5)q^n \equiv 2f_1 f_5^3 \pmod 4. \tag{4.4.107}$$

Combining (4.4.103) and (4.4.107), we get

$$\overline{C}_{5,5}^{15}(8n+5) \equiv \overline{C}_{5,5}^{15}(2n+1) \pmod 4. \tag{4.4.108}$$

Using the above relation and by induction on α, we arrive at (4.4.97).

Extracting the terms involving q^{2n+1} from (4.4.104), dividing by q and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(4n+3)q^n \equiv 2\frac{f_1^2 f_5^2 f_{10}}{f_2} \pmod 4. \tag{4.4.109}$$

Using (1.31), the above equation reduces to

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(4n+3)q^n \equiv 2f_5^4 \pmod 4. \tag{4.4.110}$$

Collecting the terms involving q^{5n+i} on both sides of (4.4.110), we find that

$$\overline{C}_{5,5}^{15}\left(4(5n+i)+3\right)q^n \equiv 0 \pmod 4, \quad i=1,2,3,4. \tag{4.4.111}$$

Extracting the terms involving q^{5n} from (4.4.110), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(20n+3)q^n \equiv 2f_1^4 \pmod 4. \tag{4.4.112}$$

Employing (1.34) into (4.4.112) and extracting the terms involving q^{5n+4} in the resultant equation, we have

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(100n+83)q^n \equiv 2f_5^4 \pmod{4}. \tag{4.4.113}$$

In the view of congruences (4.4.110) and (4.4.113), we obtain

$$\overline{C}_{5,5}^{15}(100n+83)q^n \equiv \overline{C}_{5,5}^{15}(4n+3) \pmod{4}. \tag{4.4.114}$$

Using the above relation and by induction on α, we have

$$\overline{C}_{5,5}^{15}\left(4\cdot 5^{5\alpha+5}n+\frac{2\cdot 5^{5\alpha+6}-1}{3}\right) \equiv \overline{C}_{5,5}^{15}(4n+3) \pmod{4}. \tag{4.4.115}$$

Using congruence (4.4.111) in the above equation, we get (4.4.98).

Extracting the even terms of the equation (4.4.106), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(8n+1)q^n \equiv 2\frac{f_1^2f_2f_5^2}{f_{10}} \pmod{4}. \tag{4.4.116}$$

Invoking (1.31) in (4.4.116), we deduce

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(8n+1)q^n \equiv 2f_2^2 \pmod{4}. \tag{4.4.117}$$

Congruence (4.4.94) follows by extracting the terms involving q^{2n+1} from the above equation.

Using (1.31) in (4.4.117) implies

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(8n+1)q^n \equiv 2f_1^4 \pmod{4}. \tag{4.4.118}$$

Combining (4.4.112) and (4.4.118), we obtain (4.4.95).

Substituting (1.34) into (4.4.103), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(2n+1)q^n \equiv 2f_5^3f_{25}\left(a(q^5)-q-\frac{q^2}{a(q^5)}\right) \pmod{4}. \tag{4.4.119}$$

Extracting the terms involving q^{5n+j} on both sides of (4.4.119), we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(2(5n+j)+1) \equiv 0 \pmod{4}, \quad j=3,4. \tag{4.4.120}$$

Extracting the terms involving q^{5n+1} from (4.4.119), dividing by q and replacing q^5 by q, we obtain

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(10n+3)q^n \equiv 2f_1^3 f_5 \pmod{4}. \tag{4.4.121}$$

From (4.4.105) and (4.4.121), we obtain (4.4.96).

Employing (1.34) into (4.4.121), we arrive at

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(10n+3)q^n$$
$$\equiv f_5 f_{25}^3 \left(a^3(q^5) - 3a^2(q^5)q + 5q^3 - \frac{3q^5}{a^2(q^5)} - \frac{q^6}{a^3(q^5)} \right) \pmod{4}, \tag{4.4.122}$$

which implies,

$$\overline{C}_{5,5}^{15}(10(5n+k)+3) \equiv 0 \pmod{4}, \quad k=2,4. \tag{4.4.123}$$

Extracting the terms involving q^{5n+3} from (4.4.122), dividing by q^3 and replacing q^5 by q, we have

$$\sum_{n=0}^{\infty} \overline{C}_{5,5}^{15}(50n+33)q^n \equiv 2f_1 f_5^3 \pmod{4}. \tag{4.4.124}$$

Combining (4.4.103) and (4.4.124), we get

$$\overline{C}_{5,5}^{15}(50n+33) \equiv \overline{C}_{5,5}^{15}(2n+1) \pmod{4}. \tag{4.4.125}$$

Using the above relation and by induction on α, we have

$$\overline{C}_{5,5}^{15}\left(2\cdot 5^{2\alpha+4}n + \frac{4\cdot 5^{2\alpha+4}-1}{3}\right) \equiv \overline{C}_{5,5}^{15}(2n+1) \pmod{4}. \tag{4.4.126}$$

Using (4.4.120) in (4.4.126), we obtain (4.4.99). □

Chapter 5

ℓ-REGULAR CUBIC PARTITION PAIRS

5.1 Introduction

In chapter (1), we defined the ℓ-regular cubic partition. Kim [40] has studied congruence properties of $\overline{b}(n)$, which denotes overcubic partition pairs of n and generating function is given by

$$\sum_{n=0}^{\infty} \overline{b}(n)q^n = \frac{(-q;q)_\infty^2(-q^2;q^2)_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty^2} = \frac{f_4^2}{f_1^4 f_2^2}. \tag{5.1.1}$$

Recently, Naika et al. [62] have established some new Ramanujan like congruences and infinite families of congruences modulo powers of 2 for $\overline{b}(n)$. Motivated by the above works, we study $b_l(n)$, the number of ℓ- regular cubic partition pairs and the generating function is given by

$$\sum_{n=0}^{\infty} b_\ell(n)q^n = \frac{(q^\ell;q^\ell)_\infty^2(q^{2\ell};q^{2\ell})_\infty^2}{(q;q)_\infty^2(q^2;q^2)_\infty^2} = \frac{f_\ell^2 f_{2\ell}^2}{f_1^2 f_2^2}. \tag{5.1.2}$$

5.2 Congruences for ℓ- regular cubic partition pairs

In this section, we obtain some congruences and infinite families of congruences modulo 4, 8, 27 and 81 for $b_\ell(n)$ for various values of ℓ.

Reference [51] is based on this chapter

5.2.1 Congruences modulo 4 for $b_2(n)$

Theorem 5.2.1. *For each $\alpha \geq 0$ and $n \geq 1$,*

$$b_2(18n+8) \equiv 0 \pmod{4}, \tag{5.2.1}$$

$$b_2(18n+14) \equiv 0 \pmod{4}, \tag{5.2.2}$$

$$b_2\left(2\cdot 3^{2\alpha+4}n + \frac{11\cdot 3^{2\alpha+3}-1}{4}\right) \equiv 0 \pmod{4}. \tag{5.2.3}$$

$$b_2\left(2p^{2\alpha+1}(pn+j) + \frac{10p^{2\alpha+2}-2}{8}\right) \equiv 0 \pmod{4}, \tag{5.2.4}$$

and for $n \geq 0$, $1 \leq j \leq p-1$.

Proof. Setting $\ell = 2$ in (5.1.2), we have

$$\sum_{n=0}^{\infty} b_2(n)q^n = \frac{f_4^2}{f_1^2}. \tag{5.2.5}$$

Invoking (1.31) in (5.2.5), we obtain

$$\sum_{n=0}^{\infty} b_2(n)q^n \equiv \frac{f_2^4}{f_1^2} \equiv t_2(n) \pmod{4},$$

where $t_2(n)$ is the number of ways to write n as a sum of two triangular numbers. But $t_2(n) = \frac{1}{4}s_2(8n+2)$, where $s_2(n)$ is the number of ways to write n as the sum of two squares. This gives the following: if $8n+2 = 2n_1n_2$, where

$$n_1 = \prod_{p\equiv 1 \pmod{4}} p^r, \qquad n_2 = \prod_{p\equiv 3 \pmod{4}} p^s,$$

then

$$b_2(n) \equiv \begin{cases} d_1(8n+2) - d_3(8n+3) \pmod{4} & \text{if all } s \text{ are even,} \\ 0 \pmod{4} & \text{else,} \end{cases}$$

where $d_1(n)$ is the number of divisors of n that are congruent to 1 modulo 4 and $d_3(n)$ is the number of divisors of n that are congruent to 3 modulo 4.

This implies the congruences (5.2.1), (5.2.2), (5.2.3) and (5.2.4) follow since 3 sharply divides $144n+66$, $144n+114$, $16\cdot 3^{2\alpha+4}n + 22\cdot 3^{2\alpha+3}$ and $p^{2\alpha+1}$ sharply divides

$16 \cdot p^{2\alpha+1}(pn+j) + 10 \cdot p^{2\alpha+2}$.

□

5.2.2 Infinite families of congruence modulo 27 for $b_3(n)$

Theorem 5.2.2. *For each $n \geq 1$,*

$$b_3(9n+4) \equiv 2b_3(3n+1) + 27b_3(n) \pmod{81}. \tag{5.2.6}$$

Proof. Setting $\ell = 3$ in (5.1.2), we have

$$\sum_{n=0}^{\infty} b_3(n)q^n = \frac{f_3^2 f_6^2}{f_1^2 f_2^2}. \tag{5.2.7}$$

Substituting (1.64) into (5.2.7), we obtain

$$\sum_{n=0}^{\infty} b_3(n)q^n = \frac{f_9^6 f_{18}^6}{f_3^6 f_6^6}\left(a(q^3)^2 + 2qa(q^3)b(q^3) + q^2 b(q^3)^2 + 6q^2 a(q^3) + 6q^3 b(q^3) + 9q^4\right). \tag{5.2.8}$$

Extracting the terms involving q^{3n+1} from (5.2.8), dividing q and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} b_3(3n+1)q^n = \frac{f_3^6 f_6^6}{f_1^6 f_2^6}\left(2a(q)b(q) + 9q\right). \tag{5.2.9}$$

Using (1.61) in (5.2.9), we arrive at

$$\sum_{n=0}^{\infty} b_3(3n+1)q^n = \frac{f_3^6 f_6^6}{f_1^6 f_2^6}\left(2c(q) + 13q\right). \tag{5.2.10}$$

Employing (1.65) into (5.2.10), we get

$$\sum_{n=0}^{\infty} b_3(3n+1)q^n = 2\frac{f_3^2 f_6^2}{f_1^2 f_2^2} + 27q\frac{f_3^6 f_6^6}{f_1^6 f_2^6}. \tag{5.2.11}$$

Invoking (1.31) in (5.2.11), we obtain

$$\sum_{n=0}^{\infty} b_3(3n+1)q^n \equiv 2\frac{f_3^2 f_6^2}{f_1^2 f_2^2} + 27q\frac{f_9^2 f_{18}^2}{f_3^2 f_6^2} \pmod{81}. \tag{5.2.12}$$

From (5.2.7) it follows that

$$\sum_{n=0}^{\infty} b_3(3n+1)q^n \equiv 2\sum_{n=0}^{\infty} b_3(n)q^n + 27q\sum_{n=0}^{\infty} b_3(n)q^{3n} \pmod{81}. \tag{5.2.13}$$

Extracting the terms involving q^{3n+1} from (5.2.13), dividing q and replacing q^3 by q, we get (5.2.6). □

Corollary 5.2.1. *For each $\alpha \geq 0$ and $n \geq 1$,*

$$b_3\left(3^{\alpha+1}n + \frac{3^{\alpha+1}-1}{2}\right) \equiv 2^{\alpha+1} b_3(n) \pmod{27}. \tag{5.2.14}$$

Proof. Equation (5.2.13) implies (5.2.14). □

5.2.3 Congruences modulo 8 for $b_5(n)$

Theorem 5.2.3. *For each $\alpha \geq 0$ and $n \geq 1$,*

$$b_5(8n+5) \equiv 0 \pmod{8}, \tag{5.2.15}$$

$$b_5(16n+11) \equiv 0 \pmod{8}, \tag{5.2.16}$$

$$b_5\left(2^{2\alpha+6}n + 3\cdot 2^{2\alpha+4} - 1\right) \equiv 0 \pmod{8}. \tag{5.2.17}$$

Proof. Setting $\ell = 5$ in (5.1.2), we have

$$\sum_{n=0}^{\infty} b_5(n)q^n = \frac{f_5^2 f_{10}^2}{f_1^2 f_2^2}. \tag{5.2.18}$$

Employing (1.52) into (5.2.18),we get

$$\sum_{n=0}^{\infty} b_5(n)q^n = \frac{f_8^2 f_{10}^2 f_{20}^4}{f_2^6 f_{40}^2} + 2q\frac{f_4^3 f_{10}^3 f_{20}}{f_2^7} + q^2\frac{f_4^6 f_{10}^4 f_{40}^2}{f_2^8 f_8^2 f_{20}^2}, \tag{5.2.19}$$

which implies that

$$\sum_{n=0}^{\infty} b_5(2n+1)q^n = 2\frac{f_2^3 f_5^3 f_{10}}{f_1^7}. \tag{5.2.20}$$

Invoking (1.31) in (5.2.20), we obtain

$$\sum_{n=0}^{\infty} b_5(2n+1)q^n \equiv 2\frac{f_1 f_5^3 f_{10}}{f_2} \pmod{16}. \tag{5.2.21}$$

Substituting (1.58) into (5.2.21), we arrive at

$$\sum_{n=0}^{\infty} b_5(2n+1)q^n \equiv 4q^2\frac{f_4 f_{10} f_{20}^3}{f_2} + 2f_2^2 f_{10}^2 + 12q^3\frac{f_4^4 f_{10}^2 f_{40}^2}{f_2^2 f_8^2} + 14q\frac{f_2 f_{10}^3 f_{20}}{f_4} \pmod{16}. \tag{5.2.22}$$

Extracting the terms involving q^{2n+1} from (5.2.22), dividing by q and then replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} b_5(4n+3)q^n \equiv 12q\frac{f_2^4 f_5^2 f_{20}^2}{f_1^2 f_4^2} + 14\frac{f_1 f_5^3 f_{10}}{f_2} \pmod{16}. \tag{5.2.23}$$

Using (1.31) in (5.2.23), we have

$$\sum_{n=0}^{\infty} b_5(4n+3)q^n \equiv 12q\frac{f_5^2 f_{20}^2}{f_1^2} + 14\frac{f_1 f_5^3 f_{10}}{f_2} \pmod{16}. \tag{5.2.24}$$

Substituting (1.52) and (1.58) into (5.2.24), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} b_5(4n+3)q^n \equiv& 12q\frac{f_8^2 f_{20}^6}{f_2^4 f_{40}^2} + 8q^2\frac{f_4^3 f_{10} f_{20}^3}{f_2^5} + 12q^3\frac{f_4^6 f_{10}^2 f_{40}^2}{f_2^6 f_8^2} + 12q^2\frac{f_4 f_{10} f_{20}^3}{f_2} \\ &+ 14f_2^2 f_{10}^2 + 4q^3\frac{f_4^4 f_{10}^2 f_{40}^2}{f_2^2 f_8^2} + 2q\frac{f_2 f_{10}^3 f_{20}}{f_4} \pmod{16}.\end{aligned} \tag{5.2.25}$$

Extracting the terms involving q^{2n+1} from (5.2.25), dividing q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} b_5(8n+7)q^n \equiv 12\frac{f_4^2 f_{10}^6}{f_1^4 f_{20}^2} + 12q\frac{f_2^6 f_5^2 f_{20}^2}{f_1^6 f_4^2} + 4q\frac{f_2^4 f_5^2 f_{20}^2}{f_1^2 f_4^2} + 2\frac{f_1 f_5^3 f_{10}}{f_2} \pmod{16}. \tag{5.2.26}$$

Invoking (1.31) in (5.2.26), we obtain

$$\sum_{n=0}^{\infty} b_5(8n+7)q^n \equiv 12f_2^2f_{10}^2 + 12q\frac{f_5^2f_{20}^2}{f_1^2} + 4q\frac{f_5^2f_{20}^2}{f_1^2} + 2\frac{f_1f_5^3f_{10}}{f_2} \pmod{16}, \quad (5.2.27)$$

which implies,

$$\sum_{n=0}^{\infty} b_5(8n+7)q^n \equiv 12f_2^2f_{10}^2 + 2\frac{f_1f_5^3f_{10}}{f_2} \pmod{16}, \quad (5.2.28)$$

Employing (1.58) into the second term of the above equation, we get

$$\sum_{n=0}^{\infty} b_5(8n+7)q^n$$
$$\equiv 12f_2^2f_{10}^2 + 4q^2\frac{f_4f_{10}f_{20}^3}{f_2} + 2f_2^2f_{10}^2 + 12q^3\frac{f_4^4f_{10}^2f_{40}^2}{f_2^2f_8^2} + 14q\frac{f_2f_{10}^3f_{20}}{f_4} \pmod{16}. \quad (5.2.29)$$

Extracting the terms involving q^{2n+1} from (5.2.29), dividing q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} b_5(16n+15)q^n \equiv 12q\frac{f_2^4f_5^2f_{20}^2}{f_1^2f_4^2} + 14\frac{f_1f_5^3f_{10}}{f_2} \pmod{16}. \quad (5.2.30)$$

Using the congruences (5.2.30) and (5.2.23), we can see that

$$b_5(16n+15) \equiv b_5(4n+3) \pmod{16}. \quad (5.2.31)$$

By mathematical induction on α, we obtain

$$b_5(2^{2\alpha+4}n + 2^{2\alpha+4} - 1) \equiv b_5(4n+3) \pmod{16}. \quad (5.2.32)$$

Using (5.2.16) in (5.2.32) we get (5.2.17).

Extracting the terms involving q^{2n} from (5.2.22), replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} b_5(4n+1)q^n \equiv 4q\frac{f_2f_5f_{10}^3}{f_1} + 2f_1^2f_5^2 \pmod{16}. \quad (5.2.33)$$

But

$$2f_1^2f_5^2 \equiv 2\frac{f_5^2f_2^2}{f_1^2} \pmod 4, \tag{5.2.34}$$

which implies,

$$\sum_{n=0}^{\infty} b_5(4n+1)q^n \equiv 4q\frac{f_2f_5f_{10}^3}{f_1} + 2\frac{f_2^2f_5^2}{f_1^2} \pmod 8. \tag{5.2.35}$$

Substituting (1.52) into (5.2.35), we find that

$$\sum_{n=0}^{\infty} b_5(4n+1)q^n$$
$$\equiv 4q\frac{f_8f_{10}^3f_{20}^2}{f_2f_{40}} + 4q^2\frac{f_4^3f_{10}^4f_{40}}{f_2^2f_8f_{20}} + 2\frac{f_8^2f_{20}^4}{f_2^2f_{40}^2} + 4q\frac{f_4^3f_{10}f_{20}}{f_2^3} + 2q^2\frac{f_4^6f_{10}^2f_{40}^2}{f_2^4f_8^2f_{10}^2} \pmod 8. \tag{5.2.36}$$

Extracting the terms involving q^{2n+1} from (5.2.36), dividing q and replacing q^2 by q, we get

$$\sum_{n=0}^{\infty} b_5(8n+5)q^n \equiv 4\frac{f_4f_5^3f_{10}^2}{f_1f_{20}} + 4\frac{f_2^3f_5f_{10}}{f_1^3} \pmod 8. \tag{5.2.37}$$

Using (1.31) in (5.2.37), we get

$$\sum_{n=0}^{\infty} b_5(8n+5)q^n \equiv 4\frac{f_2^2f_5f_{10}}{f_1} + 4\frac{f_2^2f_5f_{10}}{f_1} \pmod 8. \tag{5.2.38}$$

Congruence (5.2.15) can be easily obtained from (5.2.38).

Extracting the terms involving q^{2n} from (5.2.25), replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} b_5(8n+3)q^n \equiv 4q\frac{f_2f_5f_{10}^3}{f_1} + 6f_1^2f_5^2 \pmod 8,$$

which implies,

$$\sum_{n=0}^{\infty} b_5(8n+3)q^n \equiv 4q\frac{f_2f_5f_{10}^3}{f_1} + 6\frac{f_1^4f_5^2}{f_1^2} \pmod 8, \tag{5.2.39}$$

Substituting (1.52) into (5.2.39), we find that

$$\sum_{n=0}^{\infty} b_5(8n+3)q^n \equiv 4q\frac{f_8 f_{10}^3 f_{20}^2}{f_2 f_{40}} + 4q^2\frac{f_4^3 f_{10}^4 f_{40}}{f_2^2 f_8 f_{20}} + 6\frac{f_8^2 f_{20}^4}{f_2^2 f_{40}^2} + 4q\frac{f_4^3 f_{10} f_{20}}{f_2^3} + 6q^2\frac{f_4^6 f_{10}^2 f_{40}^2}{f_2^4 f_8^2 f_{20}^2} \pmod{8}. \tag{5.2.40}$$

Extracting the terms involving q^{2n+1} from (5.2.40), dividing q and replacing q^2 by q, we have

$$\sum_{n=0}^{\infty} b_5(16n+11)q^n \equiv 4\frac{f_4 f_5^3 f_{10}^2}{f_1 f_{20}} + 4\frac{f_2^3 f_5 f_{10}}{f_1^3} \pmod{8}. \tag{5.2.41}$$

Using (1.31) in (5.2.41), we get

$$\sum_{n=0}^{\infty} b_5(16n+11)q^n \equiv 4\frac{f_4 f_5 f_{10}}{f_1} + 4\frac{f_4 f_5 f_{10}}{f_1} \pmod{8}. \tag{5.2.42}$$

Congruence (5.2.16) follows from (5.2.42).

□

5.2.4 Congruences modulo 27 and 81 for $b_9(n)$

Theorem 5.2.4. *For each $\alpha \geq 0$ and $n \geq 1$,*

$$b_9(27n+25) \equiv 0 \pmod{81}, \tag{5.2.43}$$

$$b_9\left(3^{\alpha+4}n + 3^{\alpha+4} - 2\right) \equiv 0 \pmod{27}. \tag{5.2.44}$$

Proof. Setting $\ell = 9$ in (5.1.2), we have

$$\sum_{n=0}^{\infty} b_9(n)q^n = \frac{f_9^2 f_{18}^2}{f_1^2 f_2^2}. \tag{5.2.45}$$

Substituting (1.64) into (5.2.45), we find that

$$\sum_{n=0}^{\infty} b_9(n)q^n = \frac{f_9^8 f_{18}^8}{f_3^8 f_6^8}\left(a(q^3)^2 + 2qa(q^3)b(q^3) + q^2 b(q^3)^2 + 6q^2 a(q^3) + 6q^3 b(q^3) + 9q^4\right). \tag{5.2.46}$$

Extracting the terms involving q^{3n+1} from (5.2.46), dividing q and replacing q^3 by q, we have

$$\sum_{n=0}^{\infty} b_9(3n+1)q^n = \frac{f_3^8 f_6^8}{f_1^8 f_2^8}\left(2a(q)b(q)+9q\right). \tag{5.2.47}$$

Using (1.61) in (5.2.47), we obtain

$$\sum_{n=0}^{\infty} b_9(3n+1)q^n = \frac{f_3^8 f_6^8}{f_1^8 f_2^8}\left(2c(q)+13q\right). \tag{5.2.48}$$

Employing (1.65) into (5.2.48), we get

$$\sum_{n=0}^{\infty} b_9(3n+1)q^n = 2\frac{f_3^4 f_6^4}{f_1^4 f_2^4} + 27q\frac{f_3^8 f_6^8}{f_1^8 f_2^8}. \tag{5.2.49}$$

Using (1.31) in (5.2.49), we have

$$\sum_{n=0}^{\infty} b_9(3n+1)q^n \equiv 2\frac{f_3^4 f_6^4}{f_1^4 f_2^4} + 27q f_1 f_2 f_3^5 f_6^5 \pmod{243}. \tag{5.2.50}$$

Employing (1.63) and (1.66) into (5.2.50), we can see that

$$\sum_{n=0}^{\infty} b_9(3n+1)q^n$$
$$\equiv 2f_3^4 f_6^4 \sum_{n=0}^{\infty} h(n)q^n + 27q f_3^5 f_6^5 f_9 f_{18}\left(\frac{1}{x(q^3)} - q - 2q^2 x(q^3)\right) \pmod{243}, \tag{5.2.51}$$

which implies that

$$\sum_{n=0}^{\infty} b_9(9n+7)q^n \equiv 2f_1^4 f_2^4 \sum_{n=0}^{\infty} h(3n+2)q^n - 27 f_1^5 f_2^5 f_3 f_6 \pmod{243}. \tag{5.2.52}$$

Substituting (1.67) into (5.2.52), we obtain

$$\sum_{n=0}^{\infty} b_9(9n+7)q^n \equiv 36\frac{f_3^4 f_6^4}{f_1^4 f_2^4} + 162q\frac{f_3^8 f_6^8}{f_1^8 f_2^8} - 27 f_1^5 f_2^5 f_3 f_6 \pmod{243}. \tag{5.2.53}$$

It follows from (1.31), we have

$$f_1^5 f_2^5 f_3 f_6 \equiv \frac{f_3^4 f_6^4}{f_1^4 f_2^4} \pmod{9}. \tag{5.2.54}$$

In view of (5.2.54), we can express (5.2.53) as

$$\begin{aligned}\sum_{n=0}^{\infty} b_9(9n+7)q^n &\equiv 9\frac{f_3^4 f_6^4}{f_1^4 f_2^4} + 162q\frac{f_3^8 f_6^8}{f_1^8 f_2^8} \pmod{243}\\ &\equiv 9\frac{f_3^4 f_6^4}{f_1^4 f_2^4} \pmod{81}.\end{aligned} \tag{5.2.55}$$

Invoking (1.31) in (5.2.55), we get

$$\sum_{n=0}^{\infty} b_9(9n+7)q^n \equiv 9f_3 f_6 f_1^5 f_2^5 \pmod{81}. \tag{5.2.56}$$

Employing (1.63) into (5.2.56), we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} b_9(9n+7)q^n \equiv 9f_3 f_6 f_9^5 f_{18}^5 (&-5\frac{q}{x(q^3)^4} + 30\frac{q^3}{x(q^3)^2} - 15\frac{q^4}{x(q^3)}\\ &+ 30x(q^3)q^6 + 120x(q^3)^2 q^7 - 80x(q^3)^4 q^9\\ &- 32x(q^3)^5 q^{10} + \frac{1}{x(q^3)^5}) \pmod{81}.\end{aligned} \tag{5.2.57}$$

Extracting the terms involving q^{3n+2} from (5.2.57) to obtain the congruence (5.2.43).

From (5.2.50) and (5.2.55), we obtain

$$\sum_{n=0}^{\infty} b_9(9n+7)q^n \equiv 18\sum_{n=0}^{\infty} b_9(3n+1)q^n \pmod{27}. \tag{5.2.58}$$

Equating the coefficients of q^n on both sides of the above equation, we get

$$b_9(9n+7) \equiv 18b_9(3n+1) \pmod{27}. \tag{5.2.59}$$

For each $\alpha \geq 0$, we obtain by induction that

$$b_9\left(3^{\alpha+2}n + 3^{\alpha+2} - 2\right) \equiv 18^{\alpha+1} b_9(3n+1) \pmod{27}. \tag{5.2.60}$$

Using (5.2.43) in (5.2.60), we obtain (5.2.44). □

Chapter 6

(ℓ, m)-REGULAR BIPARTITION TRIPLES

6.1 Introduction

In introductory chapter, we defined the (ℓ, m)-regular bipartition functions and denoted by $B_{\ell,m}(n)$. Recently Dou [23] has discovered an infinite family of congruences modulo 11 for $B_{3,11}(n)$ and she gave several conjectures on $B_{s,t}(n)$. Xia and Yao [76] have confirmed three conjectures on $B_{3,7}(n)$ and obtained several infinite families of congruences modulo 3 and 5 for $B_{3,s}(n)$ and $B_{5,s}(n)$. In addition, also proved many infinite families of congruences modulo 7 for $B_{3,7}(n)$. Motivated by the above works, we study the function $BT_{\ell,m}(n)$, the number of (ℓ, m)-regular bipartition triples of a positive integer n. The generating function for $BT_{\ell,m}(n)$ is given by

$$\sum_{n=0}^{\infty} BT_{\ell,m}(n)q^n = \frac{f_\ell^3 f_m^3}{f_1^6}. \tag{6.1.1}$$

6.2 On (ℓ, m)-regular bipartition triples

In this section, we establish some congruences and infinite families of congruences for $BT_{\ell,m}(n)$ modulo 3, 9 and 27 for various values of ℓ and m.

Reference [53] is based on this chapter

6.2.1 Congruences modulo 3 for $BT_{2,3}(n)$

Theorem 6.2.1. *For each $n \geq 0$ and $\alpha \geq 0$,*

$$BT_{2,3}(3n+1) \equiv 0 \pmod{3}, \tag{6.2.1}$$

$$BT_{2,3}(3n+2) \equiv 0 \pmod{3}, \tag{6.2.2}$$

$$BT_{2,3}\left(3^{2\alpha+3}n + \frac{11 \cdot 3^{2\alpha+2} - 3}{8}\right) \equiv 0 \pmod{3}. \tag{6.2.3}$$

Proof. Setting $(\ell, m) = (2, 3)$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} BT_{2,3}(n)q^n = \frac{f_2^3 f_3^3}{f_1^6}. \tag{6.2.4}$$

Using (1.31) in (6.2.4), we get

$$\sum_{n=0}^{\infty} BT_{2,3}(n)q^n \equiv f_3 f_6 \pmod{3}. \tag{6.2.5}$$

Congruences (6.2.1) and (6.2.2) easily follow from (6.2.5).

From (6.2.5) yields

$$\sum_{n=0}^{\infty} BT_{2,3}(3n)q^n \equiv f_1 f_2 \pmod{3}. \tag{6.2.6}$$

Employing (1.74) in (6.2.6), we find that

$$\sum_{n=0}^{\infty} BT_{2,3}(3n)q^n \equiv \frac{f_6 f_9^4}{f_3 f_{18}^2} + 2q f_9 f_{18} + 2q^2 \frac{f_3 f_{18}^4}{f_6 f_9^2} \pmod{3}, \tag{6.2.7}$$

which yields

$$\sum_{n=0}^{\infty} BT_{2,3}(9n+3)q^n \equiv 2 f_3 f_6 \pmod{3}. \tag{6.2.8}$$

By (6.2.8) and (6.2.5), we obtain

$$BT_{2,3}(9n+3) \equiv 2BT_{2,3}(n) \pmod{3}. \tag{6.2.9}$$

Using (6.2.9) and by mathematical induction

$$BT_{2,3}\left(9^{\alpha+1}n+\frac{3\cdot 9^{\alpha+1}-3}{8}\right)\equiv 2^{\alpha+1}BT_{2,3}(n) \pmod 3. \tag{6.2.10}$$

Congruence (6.2.3) follows from (6.2.10) and (6.2.1). □

6.2.2 Congruences modulo 3 for $BT_{2,9}(n)$

Theorem 6.2.2. *For each nonnegative integer n and* $\alpha \geq 0$,

$$BT_{2,9}(3n+1)\equiv 0 \pmod 3, \tag{6.2.11}$$

$$BT_{2,9}(3n+2)\equiv 0 \pmod 3, \tag{6.2.12}$$

$$BT_{2,9}(27n+15)\equiv 0 \pmod 3, \tag{6.2.13}$$

$$BT_{2,9}(27n+24)\equiv 0 \pmod 3, \tag{6.2.14}$$

$$BT_{2,9}\left(3^{2\alpha+5}n+\frac{7\cdot 3^{2\alpha+4}-3}{4}\right)\equiv 0 \pmod 3. \tag{6.2.15}$$

Proof. Setting $(\ell, m) = (2, 9)$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} BT_{2,9}(n)q^n = \frac{f_2^3 f_9^3}{f_1^6}. \tag{6.2.16}$$

By (1.31) in (6.2.16), we get

$$\sum_{n=0}^{\infty} BT_{2,9}(n)q^n \equiv \frac{f_6 f_{18}}{f_3^2} \pmod 3. \tag{6.2.17}$$

From the above equation we obtain the congruences (6.2.11) and (6.2.12).

Equation (6.2.17) can be written as

$$\sum_{n=0}^{\infty} BT_{2,9}(3n)q^n \equiv \frac{f_2 f_6}{f_1^2} \pmod 3. \tag{6.2.18}$$

Substituting (1.72) in (6.2.18), we find that

$$\sum_{n=0}^{\infty} BT_{2,9}(3n)q^n \equiv \frac{f_6^5 f_9^6}{f_3^8 f_{18}^3}+2q\frac{f_6^4 f_9^3}{f_3^7}+q^2\frac{f_6^3 f_{18}^3}{f_3^6} \pmod 3, \tag{6.2.19}$$

which implies,

$$\sum_{n=0}^{\infty} BT_{2,9}(9n+6)q^n \equiv \frac{f_2^3 f_6^3}{f_1^6} \pmod{3}. \tag{6.2.20}$$

Using (1.31) in (6.2.20), we arrive at

$$\sum_{n=0}^{\infty} BT_{2,9}(9n+6)q^n \equiv \frac{f_6 f_{18}}{f_3^6} \pmod{3}. \tag{6.2.21}$$

Congruences (6.2.13) and (6.2.14) follow from (6.2.21).

From (6.2.21) yields

$$\sum_{n=0}^{\infty} BT_{2,9}(27n+6)q^n \equiv \frac{f_2 f_6}{f_1^6} \pmod{3}. \tag{6.2.22}$$

Using (6.2.22) and (6.2.18), we get

$$BT_{2,9}(27n+6) \equiv BT_{2,9}(3n) \pmod{3}. \tag{6.2.23}$$

Using (6.2.23) and by mathematical induction

$$BT_{2,9}\left(27^{\alpha+1}n + \frac{27\cdot 9^{\alpha}-3}{4}\right) \equiv BT_{2,9}(3n) \pmod{3}. \tag{6.2.24}$$

From (6.2.24) and (6.2.13), we find that (6.2.15). □

6.2.3 Infinite family of congruences modulo 27 for $BT_{3,3}(n)$

Theorem 6.2.3. *For each n and $\alpha \geq 0$,*

$$BT_{3,3}(3n+2) \equiv 0 \pmod{27}, \tag{6.2.25}$$

$$BT_{3,3}\left(3^{\alpha+2}n + \frac{5\cdot 3^{\alpha+1}-1}{2}\right) \equiv 0 \pmod{27}. \tag{6.2.26}$$

Proof. Setting $(\ell, m) = (3,3)$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} BT_{3,3}(n)q^n = \frac{f_3^6}{f_1^6}. \tag{6.2.27}$$

Invoking (1.31) in (6.2.27), we obtain

$$\sum_{n=0}^{\infty} BT_{3,3}(n)q^n \equiv \frac{f_1^{21}}{f_3^3} \equiv \frac{(f_1^3)^7}{f_3^3} \pmod{27}. \tag{6.2.28}$$

Employing (1.75) in (6.2.28), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} BT_{3,3}(n)q^n \equiv \frac{f_9^{21}}{f_3^3\zeta^7}&(1+6q\zeta+q^3\zeta^3+9q^4\zeta^4+12q^6\zeta^6+9q^7\zeta^7\\&+26q^9\zeta^9+12q^{10}\zeta^{10}+23q^{12}\zeta^{12}+9q^{13}\zeta^{13}+12q^{15}\zeta^{15}\\&+9q^{16}\zeta^{16}+25q^{18}\zeta^{18}+6q^{19}\zeta^{19}+22q^{21}\zeta^{21}) \pmod{27}.\end{aligned} \tag{6.2.29}$$

Congruence (6.2.25) follows from the above equation.

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (6.2.29), we get

$$\begin{aligned}&\sum_{n=0}^{\infty} BT_{3,3}(3n+1)q^n\\&\equiv \frac{f_3^{21}}{f_1^3}(6\eta^{-6}+9q\eta^{-3}+9q^2+12q^3\eta^3+9q^4\eta^6+9q^5\eta^9+6q^6\eta^{12}) \pmod{27},\end{aligned} \tag{6.2.30}$$

which implies,

$$\sum_{n=0}^{\infty} BT_{3,3}(3n+1)q^n \equiv \frac{f_3^{21}}{f_1^3}\left(6(\eta^{-1}+4q\eta^2)^6\right) \pmod{27}. \tag{6.2.31}$$

Using (1.76) in (6.2.31), we arrive at

$$\sum_{n=0}^{\infty} BT_{3,3}(3n+1)q^n \equiv \frac{f_3^{21}}{f_1^3}\left(6\left(\frac{f_1^{12}}{f_3^{12}}\right)^2\right) \pmod{27}, \tag{6.2.32}$$

which is equivalent to

$$\sum_{n=0}^{\infty} BT_{3,3}(3n+1)q^n \equiv 6\frac{f_1^{21}}{f_3^3} \pmod{27}. \tag{6.2.33}$$

From (6.2.33) and (6.2.28), we get

$$BT_{3,3}(3n+1) \equiv 6BT_{3,3}(n) \pmod{27}. \tag{6.2.34}$$

Using (6.2.34) and by mathematical induction

$$BT_{3,3}\left(3^{\alpha+1}n + \frac{3^{\alpha+1}-1}{2}\right) \equiv 6^{\alpha+1}BT_{3,3}(n) \pmod{27}. \tag{6.2.35}$$

Congruence (6.2.26) follows from (6.2.35) and (6.2.25). □

6.2.4 Congruences modulo 9 for $BT_{3,5}(n)$

Theorem 6.2.4. *For each nonnegative integer n and $\alpha \geq 0$,*

$$BT_{3,5}(5n+2) \equiv 0 \pmod{9}, \tag{6.2.36}$$

$$BT_{3,5}(5n+4) \equiv 0 \pmod{9}, \tag{6.2.37}$$

$$BT_{3,5}\left(5^{\alpha+2}n + \frac{11 \cdot 5^{\alpha+1}-3}{4}\right) \equiv 0 \pmod{9}. \tag{6.2.38}$$

Proof. Setting $(\ell, m) = (3, 5)$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} BT_{3,5}(n)q^n = \frac{f_3^3 f_5^3}{f_1^6} = \frac{f_1^3 f_3^3 f_5^3}{f_1^9}. \tag{6.2.39}$$

Using (1.31) in (6.2.39), we obtain

$$\sum_{n=0}^{\infty} BT_{3,5}(n)q^n \equiv f_1^3 f_5^3 \pmod{9}. \tag{6.2.40}$$

Substituting (1.34) in (6.2.40), we find that

$$\sum_{n=0}^{\infty} BT_{3,5}(n)q^n \equiv f_5^3 f_{25}^3 \left(a^3 + 6a^2q + 5q^3 + \frac{6q^5}{a^2} + \frac{8q^6}{a^3}\right) \pmod{9}. \tag{6.2.41}$$

Congruences (6.2.36) and (6.2.37) follow from (6.2.41).

From (6.2.41) yields

$$\sum_{n=0}^{\infty} BT_{3,5}(5n+3)q^n \equiv 5f_1^3 f_5^3 \pmod{9}. \tag{6.2.42}$$

By (6.2.42) and (6.2.40), we get

$$BT_{3,5}(5n+3) \equiv 5BT_{3,5}(n) \pmod{9}. \tag{6.2.43}$$

From (6.2.43) and by mathematical induction

$$BT_{3,5}\left(5^{\alpha+1}n + \frac{3\cdot 5^{\alpha+1}-3}{4}\right) \equiv 5^{\alpha+1}BT_{3,5}(n) \pmod{9}. \tag{6.2.44}$$

Using (6.2.44) and (6.2.36) we arrive at (6.2.38). □

6.2.5 Congruences modulo 3 for $BT_{3,7}(n)$

Theorem 6.2.5. *For each n and $\alpha \geq 0$,*

$$BT_{3,7}(3n+1) \equiv 0 \pmod{3}, \tag{6.2.45}$$

$$BT_{3,7}(3n+2) \equiv 0 \pmod{3}, \tag{6.2.46}$$

$$BT_{3,7}(12n+9) \equiv 0 \pmod{3}, \tag{6.2.47}$$

$$BT_{3,7}\left(3\cdot 4^{\alpha+2}n + 10\cdot 4^{\alpha+1} - 1\right) \equiv 0 \pmod{3}. \tag{6.2.48}$$

Proof. Setting $(\ell, m) = (3,7)$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} BT_{3,7}(n)q^n = \frac{f_3^3 f_7^3}{f_1^6}. \tag{6.2.49}$$

By (1.31) in (6.2.49), we find that

$$\sum_{n=0}^{\infty} BT_{3,7}(n)q^n \equiv f_3 f_{21} \pmod{3}. \tag{6.2.50}$$

From (6.2.50) follow the congruences (6.2.45) and (6.2.46).

Identity (6.2.50) yields

$$\sum_{n=0}^{\infty} BT_{3,7}(3n)q^n \equiv f_1 f_7 \pmod 3. \tag{6.2.51}$$

Substituting (1.55) in (6.2.51) and extracting the odd terms

$$\sum_{n=0}^{\infty} BT_{3,7}(6n+3)q^n \equiv 2f_2 f_{14} \pmod 3. \tag{6.2.52}$$

Congruence (6.2.47) follows from the above equation.

From (6.2.52) yields

$$\sum_{n=0}^{\infty} BT_{3,7}(12n+3)q^n \equiv 2f_1 f_7 \pmod 3. \tag{6.2.53}$$

Using (6.2.53) and (6.2.51), we arrive at

$$BT_{3,7}(12n+3) \equiv 2BT_{3,7}(3n) \pmod 3. \tag{6.2.54}$$

Form (6.2.54) and by mathematical induction

$$BT_{3,7}\left(12^{\alpha+1}n + 4^{\alpha+1} - 1\right) \equiv 2^{\alpha+1} BT_{3,7}(3n) \pmod 3. \tag{6.2.55}$$

By (6.2.55) and congruence (6.2.47), we obtain (6.2.48). □

6.2.6 Congruences modulo 9 and 27 for $BT_{3,9}(n)$

Theorem 6.2.6. *For each* $n \geq 0$,

$$BT_{3,9}(3n+2) \equiv 0 \pmod{27}, \tag{6.2.56}$$

$$BT_{3,9}(9n+4) \equiv 0 \pmod{27}, \tag{6.2.57}$$

$$BT_{3,9}(9n+7) \equiv 0 \pmod{27}, \tag{6.2.58}$$

$$BT_{3,9}(27n+19) \equiv 0 \pmod{27}, \tag{6.2.59}$$

$$BT_{3,9}(27n+10) \equiv 0 \pmod 9. \tag{6.2.60}$$

Proof. Setting $(\ell, m) = (3, 9)$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} BT_{3,9}(n)q^n = \frac{f_3^3 f_9^3}{f_1^6}. \tag{6.2.61}$$

Invoking (1.31) in (6.2.61), we obtain

$$\sum_{n=0}^{\infty} BT_{3,9}(n)q^n \equiv \frac{f_1^{21} f_9^3}{f_3^6} \equiv \frac{(f_1^3)^7 f_9^3}{f_3^6} \pmod{27}. \tag{6.2.62}$$

Employing (1.75) in (6.2.62), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} BT_{3,9}(n)q^n \equiv \frac{f_9^{24}}{f_3^6 \zeta^7}&(1 + 6q\zeta + q^3\zeta^3 + 9q^4\zeta^4 + 12q^6\zeta^6 + 9q^7\zeta^7 \\ &+ 26q^9\zeta^9 + 12q^{10}\zeta^{10} + 23q^{12}\zeta^{12} + 9q^{13}\zeta^{13} + 12q^{15}\zeta^{15} \\ &+ 9q^{16}\zeta^{16} + 25q^{18}\zeta^{18} + 6q^{19}\zeta^{19} + 22q^{21}\zeta^{21}) \pmod{27}. \end{aligned} \tag{6.2.63}$$

Congruence (6.2.56) follows from the above equation.

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (6.2.63), we find that

$$\begin{aligned}&\sum_{n=0}^{\infty} BT_{3,9}(3n+1)q^n \\ &\equiv \frac{f_3^{24}}{f_1^6}(6\eta^{-6} + 9q\eta^{-3} + 9q^2 + 12q^3\eta^3 + 9q^4\eta^6 + 9q^5\eta^9 + 6q^6\eta^{12}) \pmod{27},\end{aligned} \tag{6.2.64}$$

which implies that

$$\sum_{n=0}^{\infty} BT_{3,9}(3n+1)q^n \equiv \frac{f_3^{24}}{f_1^6}\left(6(\eta^{-1} + 4q\eta^2)^6\right) \pmod{27}. \tag{6.2.65}$$

Using (1.76) in (6.2.65), we arrive at

$$\sum_{n=0}^{\infty} BT_{3,9}(3n+1)q^n \equiv \frac{f_3^{24}}{f_1^6}\left(6\left(\frac{f_1^{12}}{f_3^{12}}\right)^2\right) \pmod{27}, \tag{6.2.66}$$

which is equivalent to

$$\sum_{n=0}^{\infty} BT_{3,9}(3n+1)q^n \equiv 6f_1^{18} \pmod{27}. \tag{6.2.67}$$

By (1.31) in (6.2.67), we get

$$\sum_{n=0}^{\infty} BT_{3,9}(3n+1)q^n \equiv 6f_3^6 \pmod{27}. \tag{6.2.68}$$

Congruences (6.2.57) and (6.2.58) easily follow from (6.2.68).

From (6.2.68) yields

$$\sum_{n=0}^{\infty} BT_{3,9}(9n+1)q^n \equiv 6f_1^6 \equiv 6(f_1^3)^2 \pmod{27}. \tag{6.2.69}$$

Substituting (1.75) in (6.2.69), we obtain

$$\sum_{n=0}^{\infty} BT_{3,9}(9n+1)q^n \equiv f_9^6(6\zeta^{-2} + 18q\zeta^{-1} + 21q^3\zeta + 18q^4\zeta^2 + 15q^6\zeta^4) \pmod{27}. \tag{6.2.70}$$

Congruences (6.2.59) and (6.2.60) follow from (6.2.70). □

6.2.7 Congruences modulo 9 and 27 for $BT_{9,9}(n)$

Theorem 6.2.7. *For each n and $\alpha \geq 0$,*

$$BT_{9,9}(3n+2) \equiv 0 \pmod{27}, \tag{6.2.71}$$

$$BT_{9,9}(9n+7) \equiv 0 \pmod{27}, \tag{6.2.72}$$

$$BT_{9,9}(9n+4) \equiv 0 \pmod{9}, \tag{6.2.73}$$

$$BT_{9,9}(36n+28) \equiv 0 \pmod{9}, \tag{6.2.74}$$

$$BT_{9,9}\left(9 \cdot 4^{\alpha+2}n + 30 \cdot 4^{\alpha+1} - 2\right) \equiv 0 \pmod{9}. \tag{6.2.75}$$

Proof. Setting $(\ell, m) = (9, 9)$ in (6.1.1), we have

$$\sum_{n=0}^{\infty} BT_{9,9}(n)q^n = \frac{f_9^6}{f_1^6}. \tag{6.2.76}$$

By (1.31) in (6.2.76), we get

$$\sum_{n=0}^{\infty} BT_{9,9}(n)q^n \equiv \frac{f_9^6 f_1^{21}}{f_3^9} \equiv \frac{f_9^6 (f_1^3)^7}{f_3^9} \pmod{27}. \tag{6.2.77}$$

Employing (1.75) in (6.2.77), we arrive at

$$\begin{aligned}\sum_{n=0}^{\infty} BT_{9,9}(n)q^n \equiv \frac{f_9^{27}}{f_3^9 \zeta^7}(&1 + 6q\zeta + q^3\zeta^3 + 9q^4\zeta^4 + 12q^6\zeta^6 + 9q^7\zeta^7 \\ &+ 26q^9\zeta^9 + 12q^{10}\zeta^{10} + 23q^{12}\zeta^{12} + 9q^{13}\zeta^{13} + 12q^{15}\zeta^{15} \\ &+ 9q^{16}\zeta^{16} + 25q^{18}\zeta^{18} + 6q^{19}\zeta^{19} + 22q^{21}\zeta^{21}) \pmod{27}. \end{aligned} \tag{6.2.78}$$

Congruence (6.2.71) follows from (6.2.78).

Extracting the terms containing q^{3n+1}, dividing throughout by q and then replacing q^3 by q from (6.2.78), we deduce that

$$\begin{aligned}&\sum_{n=0}^{\infty} BT_{9,9}(3n+1)q^n \\ &\equiv \frac{f_3^{27}}{f_1^9}(6\eta^{-6} + 9q\eta^{-3} + 9q^2 + 12q^3\eta^3 + 9q^4\eta^6 + 9q^5\eta^9 + 6q^6\eta^{12}) \pmod{27},\end{aligned} \tag{6.2.79}$$

which implies,

$$\sum_{n=0}^{\infty} BT_{9,9}(3n+1)q^n \equiv \frac{f_3^{27}}{f_1^9}\left(6(\eta^{-1} + 4q\eta^2)^6\right) \pmod{27}. \tag{6.2.80}$$

Using (1.76) in (6.2.80), we obtain

$$\sum_{n=0}^{\infty} BT_{9,9}(3n+1)q^n \equiv \frac{f_3^{27}}{f_1^9}\left(6\left(\frac{f_1^{12}}{f_3^{12}}\right)^2\right) \pmod{27}, \tag{6.2.81}$$

which is equivalent to

$$\sum_{n=0}^{\infty} BT_{9,9}(3n+1)q^n \equiv 6f_1^{15}f_3^3 \equiv 6(f_1^3)^5 f_3^3 \pmod{27}. \tag{6.2.82}$$

Substituting (1.75) in (6.2.82), we find that

$$\sum_{n=0}^{\infty} BT_{9,9}(3n+1)q^n \equiv f_3^3 f_9^{15}(6\zeta^{-5}+18q\zeta^{-4}+12q^3\zeta^{-2}+18q^4\zeta^{-1} + 15q^6\zeta+6q^9\zeta^4+18q^{10}\zeta^5+12q^{12}\zeta^7 + 18q^{13}\zeta^8+15q^{15}\zeta^{10}) \pmod{27}. \tag{6.2.83}$$

Congruence (6.2.72) follows from (6.2.83).

From (6.2.82) can be written as

$$\sum_{n=0}^{\infty} BT_{9,9}(3n+1)q^n \equiv 6f_3^8 \pmod{9}. \tag{6.2.84}$$

Congruence (6.2.73) easily obtained from the above equation.

From (6.2.84) yields

$$\sum_{n=0}^{\infty} BT_{9,9}(9n+1)q^n \equiv 6f_1^8 \equiv 6\frac{f_3^3}{f_1} \pmod{9}. \tag{6.2.85}$$

Employing (1.42) in (6.2.85) and extracting the odd terms

$$\sum_{n=0}^{\infty} BT_{9,9}(18n+10)q^n \equiv 6\frac{f_6^3}{f_2} \pmod{9}. \tag{6.2.86}$$

Congruence (6.2.74) follows from (6.2.86).

From (6.2.86), we have

$$\sum_{n=0}^{\infty} BT_{9,9}(36n+10)q^n \equiv 6\frac{f_3^3}{f_1} \pmod{9}. \tag{6.2.87}$$

Using (6.2.85) and (6.2.87), we obtain

$$BT_{9,9}(36n+10) \equiv BT_{9,9}(9n+1) \pmod{9}. \tag{6.2.88}$$

From (6.2.43) and by mathematical induction

$$BT_{9,9}\left(36^{\alpha+1}n+12\cdot 4^{\alpha}-2\right) \equiv BT_{9,9}(9n+1) \pmod{9}. \tag{6.2.89}$$

Using (6.2.89) and congruence (6.2.74) we get (6.2.75). □

Chapter 7

PARTITION QUADRUPLES WITH t-CORES

7.1 Introduction

In chapter (1), we have defined partition with t-cores $a_t(n)$ and partition quadruple with t-cores $C_t(n)$. Many mathematicians studied the arithmetic properties of $a_t(n)$. For instance Hirschhorn and Sellers [31,32] have studied the 4-core partition $a_4(n)$ and established some infinite families of arithmetic relations for $a_4(n)$. Baruah and Nath [8] have proved some more infinite families of arithmetic identities for $a_4(n)$. With the above motivation, we study the divisibility properties of the function $C_t(n)$.

7.2 Congruences for partition quadruples with t-cores

In this section, we obtain some congruences and infinite families of congruences for $C_t(n)$ modulo 5, 7 and 8 for various values of t.

Reference [52] is based on this chapter

7.2.1 Generating functions for $C_3(4n)$, $C_3(4n+1)$, $C_3(4n+2)$ and $C_3(4n+3)$

Theorem 7.2.1. *For each $n \geq 0$,*

$$\sum_{n=0}^{\infty} C_3(4n)q^n = \frac{f_2^{16} f_6^8}{f_1^8 f_4^4 f_{12}^4} + 24q\frac{f_2^{10} f_3^2 f_6^2}{f_1^6} + 16q^2\frac{f_2^4 f_3^4 f_4^4 f_{12}^4}{f_1^4 f_6^4} + 24q\frac{f_2^5 f_3^7 f_6}{f_1^5} + q\frac{f_3^{12}}{f_1^4}, \tag{7.2.1}$$

$$\sum_{n=0}^{\infty} C_3(4n+1)q^n = 4\frac{f_2^{12} f_3^3 f_6^6}{f_1^7 f_4^3 f_{12}^3} + 48q\frac{f_2^6 f_3^5 f_4 f_{12}}{f_1^5} + 8q\frac{f_2 f_3^{10} f_4 f_{12}}{f_1^4 f_6}, \tag{7.2.2}$$

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n = 8\frac{f_2^{13} f_3 f_6^5}{f_1^7 f_4^2 f_{12}^2} + 32q\frac{f_2^7 f_3^3 f_4^2 f_{12}^2}{f_1^5 f_6} + 6\frac{f_2^8 f_3^6 f_6^4}{f_1^6 f_4^2 f_{12}^2} + 24q\frac{f_2^2 f_3^8 f_4^2 f_{12}^2}{f_1^4 f_6^2}, \tag{7.2.3}$$

$$\sum_{n=0}^{\infty} C_3(4n+3)q^n = 24\frac{f_2^9 f_3^4 f_6^3}{f_1^6 f_4 f_{12}} + 32q\frac{f_2^3 f_3^6 f_4^3 f_{12}^3}{f_1^4 f_6^3} + 4\frac{f_2^4 f_3^9 f_6^2}{f_1^5 f_4 f_{12}}. \tag{7.2.4}$$

Proof. Setting $t = 3$ in (1.25), we have

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{f_3^{12}}{f_1^4} = \left(\frac{f_3^3}{f_1}\right)^4. \tag{7.2.5}$$

Substituting (1.42) into (7.2.5), we arrive at

$$\sum_{n=0}^{\infty} C_3(n)q^n = \frac{f_4^{12} f_6^8}{f_2^8 f_{12}^4} + 4q\frac{f_4^8 f_6^6}{f_2^6} + 6q^2\frac{f_4^4 f_6^4 f_{12}^4}{f_2^4} + 4q^3\frac{f_6^2 f_{12}^8}{f_2^2} + q^4\frac{f_{12}^{12}}{f_4^4}. \tag{7.2.6}$$

Extracting the even terms of the above equation

$$\sum_{n=0}^{\infty} C_3(2n)q^n = \frac{f_2^{12} f_3^8}{f_1^8 f_6^4} + 6q\frac{f_2^4 f_3^4 f_6^4}{f_1^4} + q^2\frac{f_6^{12}}{f_2^4}, \tag{7.2.7}$$

which yields

$$\sum_{n=0}^{\infty} C_3(2n)q^n = \frac{f_2^{12}}{f_6^4}\left(\frac{f_3^2}{f_1^2}\right)^4 + 6qf_2^4 f_6^4\left(\frac{f_3^2}{f_1^2}\right)^2 + q^2\frac{f_6^{12}}{f_2^4}. \tag{7.2.8}$$

Employing (1.47) into (7.2.8) and extracting the terms involving q^{2n} and q^{2n+1}, we get (7.2.1) and (7.2.3).

From (7.2.6), we have

$$\sum_{n=0}^{\infty} C_3(2n+1)q^n = 4\frac{f_2^8 f_3^6}{f_1^6} + 4q\frac{f_3^2 f_6^8}{f_1^2}, \tag{7.2.9}$$

which implies,

$$\sum_{n=0}^{\infty} C_3(2n+1)q^n = 4f_2^8\left(\frac{f_3^2}{f_1^2}\right)^3 + 4qf_6^8\left(\frac{f_3^2}{f_1^2}\right). \tag{7.2.10}$$

Substituting (1.47) into (7.2.10) and extracting the even and odd terms of the above equation, we obtain (7.2.2) and (7.2.4). □

7.2.2 Infinite families of congruences modulo 8 for $C_3(n)$

Theorem 7.2.2. *For each $\alpha \geq 0$ and $n \geq 0$,*

$$C_3(16n+14) \equiv 0 \pmod{8}, \tag{7.2.11}$$

$$C_3(48n+30) \equiv 0 \pmod{8}, \tag{7.2.12}$$

$$C_3\left(16^{\alpha+1}n + \frac{16\cdot 4^{\alpha}-4}{3}\right) \equiv C_3(4n) \pmod{8}. \tag{7.2.13}$$

Proof. From (7.2.3), we have

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 6\frac{f_2^8 f_3^6 f_6^4}{f_1^6 f_4^2 f_{12}^2} \pmod{8}. \tag{7.2.14}$$

Using (1.31) in (7.2.14), we get

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 6\frac{f_3^6}{f_1^6} \equiv 6\left(\frac{f_3^2}{f_1^2}\right)^3 \pmod{8}. \tag{7.2.15}$$

Employing (1.47) into (7.2.15), we find that

$$\sum_{n=0}^{\infty} C_3(4n+2)q^n \equiv 6\frac{f_4^{12} f_6^3 f_{12}^6}{f_2^{15} f_8^3 f_{24}^3} + 4q\frac{f_4^9 f_6^4 f_{12}^3}{f_2^{14} f_8 f_{24}} \pmod{8}. \tag{7.2.16}$$

Extracting the terms involving q^{2n+1} from (7.2.16), dividing by q and then replacing q^2 by q, we arrive at

$$\sum_{n=0}^{\infty} C_3(8n+6)q^n \equiv 4\frac{f_2^9 f_3^4 f_6^3}{f_1^{14} f_4 f_{12}} \pmod{8}. \tag{7.2.17}$$

Invoking (1.31) in (7.2.17), we obtain

$$\sum_{n=0}^{\infty} C_3(8n+6)q^n \equiv 4f_6^3 \pmod{8}. \tag{7.2.18}$$

Congruence (7.2.11) follows from (7.2.18).

From (7.2.18), we have

$$\sum_{n=0}^{\infty} C_3(24n+6)q^n \equiv 4f_2^3 \pmod{8}. \tag{7.2.19}$$

Congruence (7.2.12) easily follows from above equation.

From (7.2.1), we get

$$\sum_{n=0}^{\infty} C_3(4n)q^n \equiv \frac{f_2^{16} f_6^8}{f_1^8 f_4^4 f_{12}^4} + q\frac{f_3^{12}}{f_1^4} \pmod{8}. \tag{7.2.20}$$

Invoking (1.31) in (7.2.20), we find that

$$\sum_{n=0}^{\infty} C_3(4n)q^n \equiv f_2^4 + q\left(\frac{f_3^3}{f_1}\right)^4 \pmod{8}. \tag{7.2.21}$$

Substituting (1.42) into second term of (7.2.21) and extracting the odd terms of the required equation. we deduce that

$$\sum_{n=0}^{\infty} C_3(8n+4)q^n \equiv \frac{f_2^{12} f_3^8}{f_1^8 f_6^4} + 6q\frac{f_2^4 f_3^4 f_6^4}{f_1^4} + q^2\frac{f_6^{12}}{f_2^4} \pmod{8}. \tag{7.2.22}$$

Using (1.31) in (7.2.22), we get

$$\sum_{n=0}^{\infty} C_3(8n+4)q^n \equiv f_2^8 + 6qf_2^2 f_6^6 + q^2\frac{f_6^{12}}{f_2^4} \pmod{8}. \tag{7.2.23}$$

Extracting the terms involving q^{2n} from (7.2.23) and then replacing q^2 by q, we find that

$$\sum_{n=0}^{\infty} C_3(16n+4)q^n \equiv f_1^8 + q\frac{f_3^{12}}{f_1^4} \pmod{8}. \tag{7.2.24}$$

Invoking (1.31) in (7.2.24), we arrive at

$$\sum_{n=0}^{\infty} C_3(16n+4)q^n \equiv f_2^4 + q\left(\frac{f_3^3}{f_1}\right)^4 \pmod{8}. \tag{7.2.25}$$

Using (7.2.25) and (7.2.21), we get

$$C_3(16n+4) \equiv C_3(4n) \pmod{8}. \tag{7.2.26}$$

By mathematical induction on α, we obtain (7.2.13). □

Theorem 7.2.3. *For α, β and $\gamma \geq 0$,*

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n \equiv 4f_1^3 \pmod{8}, \tag{7.2.27}$$

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n + 2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 4f_7^3 \pmod{8}, \tag{7.2.28}$$

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}n + 2\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}\right)q^n \equiv 4f_5^3 \pmod{8}, \tag{7.2.29}$$

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 2\cdot 3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n \equiv 4f_3^3 \pmod{8}. \tag{7.2.30}$$

$$C_3\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n + 2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right) \equiv \begin{cases} 4 \pmod{8} & \text{if } n = k(3k+1)/2 \text{ for some } k \in Z, \\ 0 \pmod{8} & \text{otherwise,} \end{cases} \tag{7.2.31}$$

Proof. Extracting the terms involving q^{2n} from (7.2.19) and replacing q^2 by q, we obtain

$$\sum_{n=0}^{\infty} C_3(48n+6)q^n \equiv 4f_1^3 \pmod{8}. \tag{7.2.32}$$

The equation (7.2.32) is the $\alpha = \beta = \gamma = 0$ case of (7.2.27).
Let us consider the case $\beta = \gamma = 0$. Suppose that the congruence (7.2.27) holds for some integer $\alpha \geq 0$. Employing the equation (1.75) in (7.2.27) with $\beta = \gamma = 0$,

$$\sum_{n=0}^{\infty} C_3(16 \cdot 3^{2\alpha+1} n + 2 \cdot 3^{2\alpha+1}) q^n \equiv 4(f_3 + q f_9^3) \pmod{8}, \tag{7.2.33}$$

which implies,

$$\sum_{n=0}^{\infty} C_3(16 \cdot 3^{2\alpha+2} n + 2 \cdot 3^{2\alpha+3}) q^n \equiv 4 f_3^3 \pmod{8}. \tag{7.2.34}$$

Therefore

$$\sum_{n=0}^{\infty} C_3(16 \cdot 3^{2\alpha+3} n + 2 \cdot 3^{2\alpha+3}) q^n \equiv 4 f_1^3 \pmod{8}, \tag{7.2.35}$$

which implies that (7.2.27) is true for $\alpha + 1$. Hence by induction (7.2.27) is true for any non-negative integer α and $\beta = \gamma = 0$.
Let us consider the case $\gamma = 0$, suppose that the congruence (7.2.27) holds for some integer $\alpha, \beta \geq 0$. Substituting (1.34) in (7.2.27),

$$\sum_{n=0}^{\infty} C_3(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta} n + 2 \cdot 3^{2\alpha+1} \cdot 5^{2\beta}) q^n \equiv 4 f_{25}^3 \left(a - q - q^2/a\right)^3 \pmod{8}. \tag{7.2.36}$$

Extracting the terms involving q^{5n+3} from (7.2.36), we arrive at

$$\sum_{n=0}^{\infty} C_3(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+1} n + 2 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+2}) q^n \equiv 4 f_5^3 \pmod{8}, \tag{7.2.37}$$

which yields

$$\sum_{n=0}^{\infty} C_3(16 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+2} n + 2 \cdot 3^{2\alpha+1} \cdot 5^{2\beta+2}) q^n \equiv 4 f_1^3 \pmod{8}, \tag{7.2.38}$$

which implies that (7.2.27) is true for $\beta + 1$. Hence by induction (7.2.27) is true for $\alpha, \beta \geq 0$ and $\gamma = 0$.
Now, Suppose that the congruence (7.2.27) holds for some integers α, β and $\gamma \geq 0$.

Employing (1.35) in the equation (7.2.27), we find that

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)q^n$$
$$\equiv 4f_{49}^3\left(\frac{B(q^7)}{C(q^7)}-q\frac{A(q^7)}{B(q^7)}-q^2+q^5\frac{C(q^7)}{A(q^7)}\right)^3 \pmod 8. \tag{7.2.39}$$

Extracting the terms involving q^{7n+6} from (7.2.39), we get

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 4f_7^3 \pmod 8, \tag{7.2.40}$$

which prove (7.2.28). Extracting the coefficient of q^{7n} in (7.2.40), we arrive at

$$\sum_{n=0}^{\infty} C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+2\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)q^n \equiv 4f_1^3 \pmod 8, \tag{7.2.41}$$

which implies that (7.2.27) is true for $\gamma+1$. Hence, by induction (7.2.27) is true for any non-negative integers α, β and γ. This completes the proof.
Employing (1.34) in (7.2.27), we get (7.2.29).
Substituting (1.75) in (7.2.27) and then extracting q^{3n+1} and q^{3n}, we obtain (7.2.30) and (7.2.31) respectively. □

Corollary 7.2.1. *For α, β and $\gamma \geq 0$, $p \in \{30,46,62,78,94,110\}$, $q \in \{34,66\}$, $r \in \{26,42,58,74\}$ and $s \in \{22,38\}$,*

$$C_3\left(16\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+34\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 8, \tag{7.2.42}$$
$$C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}n+p\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+2}\right)\equiv 0 \pmod 8, \tag{7.2.43}$$
$$C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma+1}n+q\cdot 3^{2\alpha+1}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 8, \tag{7.2.44}$$
$$C_3\left(16\cdot 3^{2\alpha+1}\cdot 5^{2\beta+2}\cdot 7^{2\gamma}n+r\cdot 3^{2\alpha+1}\cdot 5^{2\beta+1}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 8, \tag{7.2.45}$$
$$C_3\left(16\cdot 3^{2\alpha+3}\cdot 5^{2\beta}\cdot 7^{2\gamma}n+s\cdot 3^{2\alpha+2}\cdot 5^{2\beta}\cdot 7^{2\gamma}\right)\equiv 0 \pmod 8. \tag{7.2.46}$$

7.2.3 Congruences modulo 5 for $C_5(n)$

Theorem 7.2.4. *For each $n \geq 0$,*

$$C_5(5n+3) \equiv 0 \pmod{5}, \tag{7.2.47}$$

$$C_5(5n+4) \equiv 0 \pmod{5}, \tag{7.2.48}$$

$$C_5(25n+21) \equiv 0 \pmod{5}. \tag{7.2.49}$$

Proof. Setting $t = 5$ in (1.25), we have

$$\sum_{n=0}^{\infty} C_5(n)q^n = \frac{f_5^{20}}{f_1^4}. \tag{7.2.50}$$

Using (1.31) in (7.2.50), we get

$$\sum_{n=0}^{\infty} C_5(n)q^n \equiv f_1 f_5^{19} \pmod{5}. \tag{7.2.51}$$

Substituting (1.34) into (7.2.51), we find that

$$\sum_{n=0}^{\infty} C_5(n)q^n \equiv f_5^{19} f_{25}\left(a - q - \frac{q^2}{a}\right) \pmod{5}. \tag{7.2.52}$$

Congruences (7.2.47) and (7.2.48) follow from (7.2.52).

Extracting the terms involving q^{5n+1} from (7.2.52), dividing by q and then replacing q^5 by q, we get

$$\sum_{n=0}^{\infty} C_5(5n+1)q^n \equiv 4f_1^{19} f_5 \pmod{5}. \tag{7.2.53}$$

Invoking (1.31) in (7.2.53), we obtain

$$\sum_{n=0}^{\infty} C_5(5n+1)q^n \equiv 4f_1^4 f_5^4 \pmod{5}. \tag{7.2.54}$$

Again substituting (1.34) into (7.2.54), we find that

$$\sum_{n=0}^{\infty} C_5(5n+1)q^n \equiv 4a^4 f_5^4 f_{25}^4 + 4a^3 q f_5^4 f_{25}^4 + 2aq^3 f_5^4 f_{25}^4 + \frac{3q^5 f_5^4 f_{25}^4}{a} + \frac{3q^6 f_5^4 f_{25}^4}{a^2} + 3a^2q^2 f_5^4 f_{25}^4 + \frac{q^7 f_5^4 f_{25}^4}{a^3} + \frac{4q^8 f_5^4 f_{25}^4}{a^4} \pmod 5. \tag{7.2.55}$$

Congruence (7.2.49) easily follows from (7.2.55). □

7.2.4 Congruences modulo 7 for $C_7(n)$

Theorem 7.2.5. *For each $n \geq 0$,*

$$C_7(7n+6) \equiv 0 \pmod 7. \tag{7.2.56}$$

Proof. Setting $t = 7$ in (1.25), we have

$$\sum_{n=0}^{\infty} C_7(n)q^n = \frac{f_7^{28}}{f_1^4}. \tag{7.2.57}$$

Invoking (1.31) in (7.2.57), we get

$$\sum_{n=0}^{\infty} C_7(n)q^n \equiv f_1^3 f_7^{27} \pmod 7. \tag{7.2.58}$$

Employing (1.35) into (7.2.58), we deduce that

$$\sum_{n=0}^{\infty} C_7(n)q^n \equiv f_7^{27} f_{49}^3 \frac{B(q^7)^3}{C(q^7)^3} + 4q f_7^{27} f_{49}^3 \frac{B(q^7)A(q^7)}{C(q^7)^2} + 3q^5 f_7^{27} f_{49}^3 \frac{B(q^7)^2}{C(q^7)A(q^7)} + 3q^2 f_7^{27} f_{49}^3 \frac{A(q^7)^2}{B(q^7)C(q^7)} + 6q^3 f_7^{27} f_{49}^3 \frac{A(q^7)}{C(q^7)} + q^7 f_7^{27} f_{49}^3 \frac{B(q^7)}{A(q^7)} + 3q^{10} f_7^{27} f_{49}^3 \frac{B(q^7)C(q^7)}{A(q^7)^2} + 3q^7 f_7^{27} f_{49}^3 \frac{A(q^7)C(q^7)}{B(q^7)^2}$$

$$
\begin{aligned}
&+6q^8 f_7^{27} f_{49}^3 \frac{C(q^7)}{B(q^7)} + 4q^{11} f_7^{27} f_{49}^3 \frac{C(q^7)^2}{A(q^7)B(q^7)} \\
&+3q^9 f_7^{27} f_{49}^3 \frac{C(q^7)}{A(q^7)} + 4q^{12} f_7^{27} f_{49}^3 \frac{C(q^7)^2}{A(q^7)^2} \\
&+q^{15} f_7^{27} f_{49}^3 \frac{C(q^7)^3}{A(q^7)^3} + 4q^2 f_7^{27} f_{49}^3 \frac{B(q^7)^2}{C(q^7)^2} \\
&+3q^4 f_7^{27} f_{49}^3 \frac{B(q^7)}{C(q^7)} + 6q^3 f_7^{27} f_{49}^3 \frac{A(q^7)^3}{B(q^7)^3} \\
&+4q^4 f_7^{27} f_{49}^3 \frac{A(q^7)^2}{B(q^7)^2} + 4q^5 f_7^{27} f_{49}^3 \frac{A(q^7)}{B(q^7)} \pmod 7. \qquad (7.2.59)
\end{aligned}
$$

Congruence (7.2.56) follows from (7.2.59). □

7.2.5 Congruences modulo 5 for $C_{25}(n)$

Theorem 7.2.6. *For each $n \geq 0$,*

$$C_{25}(5n+3) \equiv 0 \pmod 5, \qquad (7.2.60)$$

$$C_{25}(5n+4) \equiv 0 \pmod 5, \qquad (7.2.61)$$

$$C_{25}(25n+21) \equiv 0 \pmod 5. \qquad (7.2.62)$$

Proof. Setting $t = 25$ in (1.25), we have

$$\sum_{n=0}^{\infty} C_{25}(n)q^n = \frac{f_{25}^{100}}{f_1^4}. \qquad (7.2.63)$$

Using (1.31) in (7.2.50), we find that

$$\sum_{n=0}^{\infty} C_{25}(n)q^n \equiv \frac{f_1 f_{25}^{100}}{f_5} \pmod 5. \qquad (7.2.64)$$

The rest of the proof is similar to the theorem (7.2.4) therefore we omitte the details. □

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